THE ELLSBERG PARADOX AND RISK AVERSION: AN ANTICIPATED UTILITY APPROACH*

by

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Abstract

This paper describes a decision process under which it is rational to prefer a lottery with known probabilities to a similar ambiguous lottery where the decision maker does not know the exact values of the probabilities (the "Ellsberg paradox"). This is done by modelling an ambiguous lottery as a two-stage lottery, by assuming the independence axiom without the reduction of compound lotteries axiom, and by using the Anticipated Utility (rather than the Expected Utility) functional. This paper also gives conditions under which a less ambiguous lottery is preferred to a more ambiguous one and presents some comparative statics analysis, as well as some interpersonal comparisons. Finally, the connection between risk averion and ambiguity aversion and between the Ellsberg paradox and other non-expected utility phenomena are discussed, and it is proved that within the Anticipated Utility framework, risk aversion and ambiguity aversion are almost identical. This is in contrast with other theories, where ambiguity (or uncertainty) and risk are treated as different concepts.

1. Introduction

The Ellsberg paradox has puzzled economists and psychologists since its presentation about twenty-five years ago (Ellsberg (1961)). The traditional analysis of decision making under uncertainty assumed that the decision maker's preferences on prizes (represented by his cardinal utility function), together with his belief relation on events (represented by his subjective probability function) uniquely define his preferences on lotteries (Ramsey (1931), von Neumann and Morgenstern (1947), Savage (1954)). Non-Expected Utility theories such as Prospect Theory (Kahneman and Tversky (1979)), Machina's functional (1982), Chew's Weighted Utility (1983), and Anticipated Utility Theory (Quiggin (1982), Yaari (1984), Segal (1984)) also assume that a lottery is fully characterized by its possible prizes and their corresponding probabilities. Ellsberg's problems suggested situations needing more information. Consider the following problems.

Problem 1: Urn I contains 100 balls, either red (R) or black (B), with unknown numbers of each. Urn II contains 50 red balls and 50 black balls. In lottery C_i one ball is drawn at random out of urn i, i = I or II, and the player receives \$100 if the color is C, C = R or B. Ellsberg predicted, and it was confirmed later by others (Becker and Brownson (1964), MacCrimmon (1965), MacCrimmon and Larsson (1979)), that most people are indifferent to the choice between R_I and B_I and also to the choice between R_{II} and B_{II}, but they prefer R_{II} to R_I and B_{II} to B_I These preferences seem to contradict themselves, as the equivalence of R_1 and B_1 implies that the decision maker believes the probabilities of both colors to be $\frac{1}{2}$, but the preferences of R_{II} to R_I and of B_{II} to B_I suggest that the probabilities of R_I and B_I are both less than $\frac{1}{2}$.

Problem 2: An urn contains 90 balls, of which 30 are red (R). The other 60 are either black (B) or yellow (Y), but of unknown numbers of each. One ball will be drawn at random from this urn. Consider the following four lotteries.

A_1: Receive \$100 if R, otherwise nothing.

A_2: Receive \$100 if B, otherwise nothing.

B_1: Receive \$100 if R or Y, otherwise nothing.

B_2: Receive \$100 if B or Y, otherwise nothing.

Ellsberg predicted that most people will prefer A_1 to A_2, but B_2 to B_1. Empirical evidence gathered by Slovik and Tversky (1975) and MacCrimmon and Larsson (1979) supports this prediction. Ellsberg claimed that such behavior necessarily contradicts Savage's Sure Thing Principle.

Sure Thing Principle: Let f, f', g, and g' be lotteries and let S be an event. If on S f = g and f' = g', and on ~S (not S) f = f' and g = g', then f is preferred to f' if and only if g is preferred to g' (Savage (1954)).

The common response to Problem 2 contradicts this axiom. To see this, let $S = R \cup B$, $f = A_1$, $f' = A_2$, $g = B_1$, and $g' = B_2$ (see Table 1).

	30	60	
	R	B	Y
A_1	100	0	0
A_2	0	100	0
B_1	100	0	100
B_2	0	100	100

Table 1

Problem 2 presents behavioral patterns inconsistent with the Sure Thing Principle, but these patterns also seem to contradict a more fundamental decision rule known as First Order Stochastic Dominance.

First Order Stochastic Dominance: Let $F_A(x)$ be the probability of winning not more than x in lottery A. If for every x, $F_A(x) \leq F_B(x)$, then lottery A is (weakly) preferred to lottery B.

By this axiom, if x is a desired outcome, then the lottery $(x,S;0,\sim S)$ (i.e., x if S happens, otherwise nothing) is preferred to the lottery $(x,T;0,\sim T)$ if and only if the (subjective) probability of S is greater than that of T. In Problem 2, $P(R) = \frac{1}{3}$, $P(B \cup Y) = \frac{2}{3}$. Choosing A_1 over A_2 implies that $\frac{1}{3} = P(R) > P(B)$, hence $P(R \cup Y) = 1 - P(B) > \frac{2}{3} = P(B \cup Y)$. Preferring B_2 to B_1 thus contradicts the First Order Stochastic Dominance axiom.

FOSD is probably the most acceptable axiom in analyzing decision makers behavior under uncertainty. Some theories, such as Machina (1982), Quiggin (1982), Chew (1983), Yaari (1984), and Segal (1984), do not accept the Sure Thing Principle, but they all accept the First Order Stochastic Dominance axiom. The Ellsberg paradox thus challenges not only Expected Utility Theory, but every other theory of rational behavior under uncertainty in which probabilities are additive. (For models with nonadditive probabilities, see Fishburn (1983), Schmeidler (1984), Einhorn and Hogarth (1985), and Section 9 below.)

Ambiguous probabilities (i.e., situations where decision makers do not know the exact values of the probabilities) has some clear economic relevance. In some insurance problems, decision makers are less informed than the insurance companies, hence, even if the insurance company regards each policy as a well-defined lottery, consumers may regard the same insurance policies as ambiguous lotteries. For example, the same car insurance policy may be considered by the insurance company as the lottery "pay \$10,000 with probability 0.05, otherwise nothing," while the insurant considers it as "receive \$10,000 if I go into an accident, otherwise nothing," where the probability of the event "going into an accident" is between 0 and 0.1. Moreover, even well-defined probabilities may become ambiguous over time (see Einhorn and Hogarth (1984)). Other situations where ambiguity of the probabilities may play a significant role occur in search problems or in optimal investment problems. In all these cases, decision makers have some information about the objective probabilities, but they do not know their exact values.

This paper suggests that the ambiguous lottery $(x,S;0,\sim S)$ (ambiguous in the sense that the decision maker does not know the probability of S) should be considered a two-stage lottery, where the first, imaginary, stage is over the possible values of the probability of S. Denote the outcome of this first lottery by $\tilde{p}$ and its mean value by $\bar{p}$. (In Problem 1, $\bar{p} = \frac{1}{2}$; in Problem 2-A_2, $\bar{p} = \frac{1}{3}$; and in Problem 2-A_3, $\bar{p} = \frac{2}{3}$.) For this imaginary lottery it is assumed that the decision maker considers the urn of unknown composition as though sampled from a set of urns. The outcome of this stage depends on the decision maker's beliefs concerning this sampling process. In the second stage, the decision maker participates in the lottery $(x,\tilde{p};0,1-\tilde{p})$ yielding x with probability $\tilde{p}$ and 0 with probability $1-\tilde{p}$. Of course, the decision maker does not know the exact value of $\tilde{p}$, the probability of the event S, but it is assumed that he has a (subjective) probability distribution over its possible values. Consider the following example.

Suppose that in Problem 1-I he chose R. He may assume that the probability that the number of red balls is 30 $(\tilde{p} = \frac{3}{10})$ equals the probability that the number of black balls is 30 $(\tilde{p} = \frac{7}{10})$, and that these two probabilities equal $\frac{1}{5}$, while the probability of an even distribution $(\tilde{p} = \frac{1}{2})$ is $\frac{3}{5}$. The expected value of this distribution of $\tilde{p}$, denoted by $\bar{p}$, is $\frac{1}{2}$. This distribution is symmetric around its mean, as is reasonable to assume in this problem. If the decision maker uses the Reduction of Compound Lotteries Axiom, namely, if he calculates the simple probabilities of winning the various prizes by multiplying the corresponding probabilities, then this lottery immediately reduces to the simple lottery $(100,\frac{1}{2};0,\frac{1}{2})$, which is the same as the lottery in Problem 1-II. It is therefore an essential assumption of the approach developed in this paper that the decision maker does not use this reduction assumption. An immediate consequence of this observation is that this approach cannot make use of Expected Utility Theory, as this theory necessarily assumes the Reduction of Compound Lotteries Axiom.

Instead of Expected Utility Theory, I use here Anticipated Utility Theory (first suggested by Quiggin (1982), but see also Yaari (1984) and Segal (1984, 1985)), which has proved itself useful in unravelling some well known paradoxes, as the Allais paradox and the common ratio effect (see Section 5 below). It should be emphasized that modelling the Ellsberg paradox as a two stage lottery does not depend on Anticipated Utility Theory, but on the existence of a theory which does not necessarily satisfy the Reduction of Compound Lotteries Axiom. Other theories, like Machina's functional (1982) or Chew's Weighted Utility (1983) may also be used. However, the detailed results of this paper depend on the Anticipated Utility Functional.

The assumption that decision makers do not satisfy the Reduction of Compound Lotteries Axiom is not straightforward. de Finetti (1937) proved that disobeying this axiom makes the decision maker vulnerable to Dutch books, i.e., he may find himself willing to pay a certain positive amount of money, without moving from his initial position. (For this claim see also Freedman and Purves (1969) and Yaari (1985b). See also the exchange of ideas about "probabilities of probabilities" in Marschak (1975)). However, this argument holds only if the second stage of the compound lottery is conducted right after the first stage, (but, in this case one can hardly regard the lottery as a real two-stage lottery). On the other hand, if sufficiently long time passes between the two stages of the lottery, then there is no reason to make this reduction assumption (see, for example, Kreps and Porteos (1978)). Indeed, empirical evidence show that decision makers do not necessarily obey this axiom (see Kahneman and Tversky (1979), Snowball and Brown (1979), and especially Ronen (1971) where the two stages of the compound lottery are separated). It is my belief that decision makers consider the Ellsberg urn as a real two-stage lottery, in which the first, imaginary, stage and the second, real, stage are clearly distinguishable. Therefore, they do not feel themselves obliged to obey the Reduction of Compound Lotteries Axiom, in the same way they do not obey it when the two stages are separated by a sufficiently long period of time.

Ambiguous lotteries appear to be riskier than "clear" lotteries where the probabilities are well-defined, and are known to the decision maker. In the numerical example concerning Problem 1-I, $\tilde{p}$ is a random variable with expectation $\frac{1}{2}$, hence a risk-averse decision maker will prefer the "certain" probability $\frac{1}{2}$ to the random probability with expected value $\frac{1}{2}$. This argument, although intuitive, should not be used in this way, because the Expected Utility definition of risk aversion deals with uncertain prizes and not with uncertain probabilities. Therefore, one cannot use the traditional concept of risk aversion, which is equivalent to the assumption of diminishing marginal utility of wealth, to determine that known probabilities are preferred to random probabilities, although one may define a new, independent concept of risk aversion for such situations. Indeed, most writers in this area, including Ellsberg himself, suggested a distinction between ambiguity (or uncertainty) and risk. One of the aims of this paper is to show that (at least within the Anticipated Utility framework) there is no real difference between these concepts.

The concept of risk aversion was extended to Anticipated Utility Theory in several recent works (Yaari (1985a), Chew, Karni, and Safra (1985)). One of the aims of this paper is to show that in the Anticipated Utility framework, the conditions for risk aversion and for dislikeness of ambiguity are almost the same. In other words, in the Anticipated Utility model, risk aversion and ambiguity aversion are two sides of the same coin, and the rejection of the Ellsberg urn does not require a new concept of ambiguity aversion, as was suggested by Ellsberg himself, nor a new concept of risk aversion, as suggested in the preceding paragraph. Moreover, the same conditions on the Anticipated Utility functional are needed to solve the Ellsberg paradox and some other paradoxes, as the Allais paradox and the common ratio effect. This subject is discussed in Section 5 below.

The rest of the paper is organized as follows. Section 2 describes Anticipated Utility Theory (Quiggin (1982), Yaari (1984), Segal (1984)) with two-stage lotteries (Segal (1984)). In Section 3 I calculate the value of an ambiguous lottery by using the value functions of Section 2. Section 4 discusses conditions under which the clear lottery $(x,\bar{p};0,1-\bar{p})$ is preferred to the ambiguous lottery $(x,S;0,\sim S)$. Section 6 presents the order "more ambiguous than," and conditions under which a less ambiguous lottery is preferred to a more ambiguous one. Section 7 shows some comparative statics analysis, and Section 8 compares the behavior of two decision makers who differ only in their decision-weights function (this function is defined in

Section 2). Section 9 completes the paper with a short survey of the relevant literature.

2. Anticipated Utility and the Independence Axiom

This section briefly describes Anticipated Utility Theory for one- and two-stage lotteries. Let L_1 be the family of all the bounded random varibles over R. For every $A \varepsilon L_1$, define the cumulative distribution function F_A by $F_A(x) = Pr(A \leq x)$. Let $A^+ = \inf \{x: F_A(x) = 1\}$ and let $A^- = \sup \{x: F_A(x) = 0\}$.

Let L_1^* be the set of all the elements of L_1 for which the range of F_A is finite. Elements of L_1^*, called prospects, are denoted by vectors of the form $(x_1,p_1;\ldots;x_n,p_n)$, where $x_1 \leq \ldots \leq x_n$ and $\Sigma p_i = 1$. Such a vector represents a lottery yielding x_i dollars with probability p_i, $i = 1,\ldots,n$. Obviously, if $A = (x_1,p_1;\ldots;x_n,p_n)$, then

$$F_A(x) = \begin{cases} 0 & x < x_1 \\ \sum_{j=1}^{i} p_j & x_i \leq x < x_{i+1} \\ 1 & x \geq x_n \end{cases}$$

On L_1 assume the existence of a complete and transitive binary relation, $\succsim$. $A \sim B$ iff $A \succsim B$ and $B \succsim A$, and $A \succ B$ iff $A \succsim B$ but not $B \succsim A$. The function $V: L_1 \to R$ represents the relation $\succsim$ if for every $A,B \varepsilon L_1$, $A \succsim B$ iff $V(A) \geq V(B)$. The most famous example for such a representation is the expected utility functional, given by

(2.1) $$V(A) = \int_{-\infty}^{\infty} u(x)dF_A(x)$$

On L_1^*, this function is reduced to

(2.2) $$V(x_1,p_1;\ldots;x_n,p_n) = \Sigma p_i u(x_i)$$

This theory, despite its simplicity and its normative appeal, fails to explain some well known evidence, as the Allais paradox or the common ratio effect (Allais (1953), MacCrimmon and Larsson (1979). For a presentation of these phenomena see Section 5 below). In recent years several alternatives have emerged to Expected Utility Theory (Kahneman and Tversky (1979), Machina (1982), Chew (1983)). Quiggin (1982) suggested the following generalization of (2.1)-(2.2), called Anticipated Utility

$$(2.3) \qquad V(A) = u(A^-) + \int_{A^-}^{A^+} u'(x)f(1-F_A(x))dx = \int_{A^-}^{A^+} u(x)df(1-F_A(x))$$

where the decision-weights function $f: [0,1] \to [0,1]$ satisfies $f(0) = 0$ and $f(1) = 1$.[1] On L_1^* (2.3) takes one of the following equivalent forms

$$(2.4) \quad V(x_1,p_1;\ldots;x_n,p_n) = u(x_n)f(p_n) + \sum_{i=1}^{n-1} u(x_i)[f(\sum_{j=i}^{n} p_j) - f(\sum_{j=i+1}^{n} p_j)] =$$

$$(2.5) \qquad u(x_1) + \sum_{i=2}^{n} [u(x_i) - u(x_{i-1})]f(\sum_{j=i}^{n} p_j)$$

Hereafter assume that $u(0) = 0$. Note that when f is linear, (2.3)-(2.5) are reduced to the Expected Utility functional (2.1)-(2.2).

To illustrate the difference between Expected Utility Theory and Anticipated Utility Theory consider Figure 1. The lottery A can be represented either by the graph of the function F_A, or by the set $A^o = \{(x,p) \in \mathbf{R} \times [0,1]: F_A(x) \leq p \leq 1\}$. Both theories agree that the value of the lottery A is the measure of the set A^o, which is a product measure of two

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[1]Quiggin suggested some additional restrictions on this function, obtained from his axioms (e.g., $f(\frac{1}{2}) = \frac{1}{2}$). Segal (1985) shows that some empirical evidence suggests a (strictly) convex f, hence $f(\frac{1}{2}) < \frac{1}{2}$. For a discussion of the properties of the function f see Section 5 below.

measures, one on the prizes' axis, the other on the probabilities' axis. By Expected Utility the measure on the probabilities is the Lebesgues' measure, while Anticipated Utility Theory allows for more general measures (by this theory, the measure of the segment $[p,q]$ is $f(q)-f(p)$). Expected Utility Theory thus becomes a special case of Anticipated utility Theory.

Equivalently to Expected Utility Theory, one may assume a general measure on the probabilities, and a linear measure on the prizes. This dual approach is discussed by Yaari (1984).

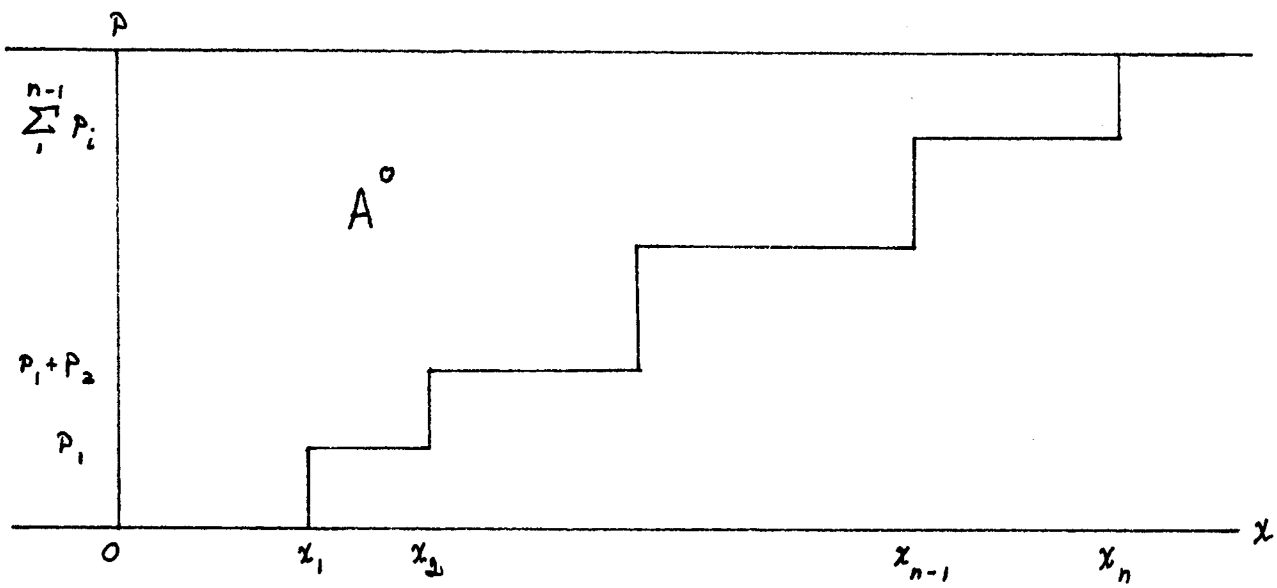

Figure 1

Let $L_2^* = \{(A_1,p_1;\ldots;A_m,p_m);\ \ \Sigma p_i = 1,\ \ p_1,\ldots,p_m > 0,\ \ A_1,\ldots,A_m \in L_1^*\}$. Elements of L_2, called two-stage lotteries, are denoted by X, Y, etc. A lottery $X \in L_2^*$ yields a ticket to lottery A_i with probability p_i, $i = 1,\ldots,m$. More specifically, at time t_1 the decision maker faces the lottery $(1,p_1;\ldots;m,p_m)$. Upon winning the number i, he participates at time $t_2 > t_1$ in the lottery A_i. It is assumed that the decision maker's discount rate for future income is 1. Thus, once he knows that he won a certain amount of money, the actual time at which he receives this prize does not make any difference to him.

Let $\succsim_2$ be a complete and transitive preference relation on L_2. The decision maker is time neutral, thus L_1^* naturally becomes isomorphic to a subspace of L_2^*, where $(x_1,p_1;\ldots;x_n,p_n)$ and

$$((x_1,1),p_1;\ldots;(x_n,1),p_n)$$

are equally attractive. The subscript 2 is therefore omitted and the preference relation over one- and two-stage lotteries is denoted by $\succsim$. A similar discussion holds for mixed lotteries, where the set of prizes is $\mathbb{R} \cup L_1^*$.

This last discussion is relevant for lotteries of the form

$$((x_1,1),p_1;\ldots;(x_n,1),p_n)$$

only. So far nothing restricts the decision maker in comparing other lotteries in L_2^* with lotteries in L_1. The following two axioms deal with such comparisons.

2.1. <u>Reduction of Compound Lotteries Axiom</u> (RCLA): If the decision maker is indifferent to the resolution timing of the uncertainty, then he may assume both stages to be conducted at time t_1. Thus, a two-stage lottery is reduced to a simple one-stage lottery. Formally, let $A_i = (x_1^i,p_1^i;\ldots;x_{n_i}^i,p_{n_i}^i)$, $i = 1,\ldots,m$.

(2.6) $$(A_1,p_1;\ldots;A_m,p_m) \sim$$
$$(x_1^1,p_1p_1^1;\ldots;x_{n_1}^1,p_1p_{n_1}^1;\ldots;x_1^m,p_mp_1^m;\ldots;x_{n_m}^m,p_mp_{n_m}^m)$$

2.2. <u>Independence Axiom</u> (IA): The relation $\succsim$ on L_2^* induces several relations on L_1^*. The independence axiom assumes that these relations coincide and are equal to $\succsim$ on L_1^*. Formally, (2.7) $$(A_1,p_1;\ldots;B,p_i;\ldots;A_m,p_m) \succsim (A_1,p_1;\ldots;C,p_i;\ldots;A_m,p_m) \Leftrightarrow B \succsim C.$$

(2.1)-(2.2) are the only continuous functions satisfying both (2.6) and (2.7). Anticipated Utility is compatible with RCLA or IA. Some empirical evidence concerning two-stage lotteries suggest that decision makers accept IA, but not necessarily RCLA. Consider the following problems, taken from Kahneman and Tversky (1979).

(a) Choose between

$$A_1 = (3000,1) \quad \text{and} \quad A_2 = (0,0.2;4000,0.8)$$

(b) Choose between

$$B_1 = (0,0.75;3000,0.25) \quad \text{and} \quad B_2 = (0,0.2;4000,0.8)$$

(c) Choose between

$$X_1 = (0,0.75;A_1,0.25) \quad \text{and} \quad X_2 = (0,0.75;A_2,0.25)$$

By IA, $X_1 \succsim X_2$ iff $A_1 \succsim A_2$, while by RCLA $X_1 \succsim X_2$ iff $B_1 \succsim B_2$. Kahneman and Tversky found that most people prefer A_1 to A_2, B_2 to B_1, and X_1 to X_2. For other empirical evidence proving that decision makers do not obey RCLA see Ronen (1971) and Snowball and Brown (1979).

Let $CE(A)$ be the certainty equivalence of A, given implicitly by $(CE(A),1) \sim A$. If $\succsim$ satisfies IA, then

(2.8) $$(A_1,p_1;\ldots;A_m,p_m) \sim (CE(A_1)p_1;\ldots;CE(A_m),p_m)$$

If $\succsim$ can be represented by the Anticipated Utility Function of (2.4)-(2.5), then $CE(A) = u^{-1}(V(A))$. Let $(A_1,p_1;\ldots;A_m,p_m) \in L_2^*$ and assume, without loss of generality, that $CE(A_1) < \ldots < CE(A_m)$. (2.8) thus implies that

(2.9) $$(A_1,p_1;\ldots;A_m,p_m) \sim (u^{-1}(V(A_1)),p_1;\ldots;u^{-1}(V(A_m)),p_m)$$

(2.9) holds of course for Expected Utility Theory, being a special case of Anticipated Utility Theory. If $\succsim$ satisfies IA and can be represented by

(2.2) (the expected utility functional), then by (2.9)

$$\text{(2.10)} \qquad V(A_1,p_1;\ldots;A_m,p_m) = \sum_{i=1}^{m} p_i V(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} p_i p_j^i u(x_j^i)$$

On the other hand, if $\succsim$ satisfies IA and can be represented by (2.4)-(2.5) (the anticipated utility functional), then by (2.9)

$$\text{(2.11)} \qquad V(A_1,p_1;\ldots;A_m,p_m) = V(A_1) + \sum_{i=2}^{m} [V(A_i) - V(A_{i-1})] f(\sum_{j=i}^{m} p_j) =$$

$$u(x_1^1) + \sum_{k=2}^{n_1} [u(x_k^1) - u(x_{k-1}^1)] f(\sum_{\ell=k}^{n_1} p_\ell^1) +$$

$$\sum_{i=2}^{m} \{u(x_1^i) + \sum_{k=2}^{n_i} [u(x_k^i) - u(x_{k-1}^i)] f(\sum_{\ell=k}^{n_i} p_\ell^i) -$$

$$u(x_1^{i-1}) - \sum_{k=2}^{n_{i-1}} [u(x_k^{i-1}) - u(x_{k-1}^{i-1})] f(\sum_{\ell=k}^{n_{i-1}} p_\ell^{i-1})\} \; f(\sum_{j=i}^{m} p_j).$$

This formula can be extended for cases where the two stage lottery is a nondiscreet distribution over lotteries, as in the case in the next section. From now on I assume that the decision maker is an Anticipated Utility maximizer, that is, his value function is given by (2.3)-(2.5).

3. The Value of an Ambiguous Lottery

This section discusses how decision makers evaluate ambiguous lotteries, that is, lotteries in which they do not know the exact values of the probabilities. Suppose that a decision maker participates in the lottery $(x,S;0,\sim S)$, where $x > 0$. In this lottery he will get x dollars if S happens and 0 otherwise. If he knows that the probability of S equals $\bar{p}$, then by (2.4) the value of this lottery is $u(x)f(\bar{p})$ (as before, $u(0) = 0$). Suppose, however, that the decision maker does not know the exact probability of S, but rather has some beliefs on the possible values of this probability. These beliefs may be discrete, in which case $P^*(p)$ denotes the probability that the probability of S is p, or they may be nondiscrete, in which case $F^*(p)$ denotes the probability that the probability of S is not greater than p. It is assumed that when confronted with the ambiguous lottery $(x,S;0,\sim S)$, the decision maker considers it a two-stage lottery. The first stage is over the random variable p (with distribution function F^*) and its outcome is denoted by $\tilde{p}$. In the second stage, the decision maker participates in the lottery $(x,\tilde{p};0,1-\tilde{p})$. For example, if F^* is discrete and P^* is given by $P^*(\bar{p}-\xi) = P^*(\bar{p}+\xi) = \alpha$, $P^*(\bar{p}) = 1 - 2\alpha$, then the two-stage representation of the ambiguous lottery $(x,S;0,\sim S)$ is as shown in Figure 2.

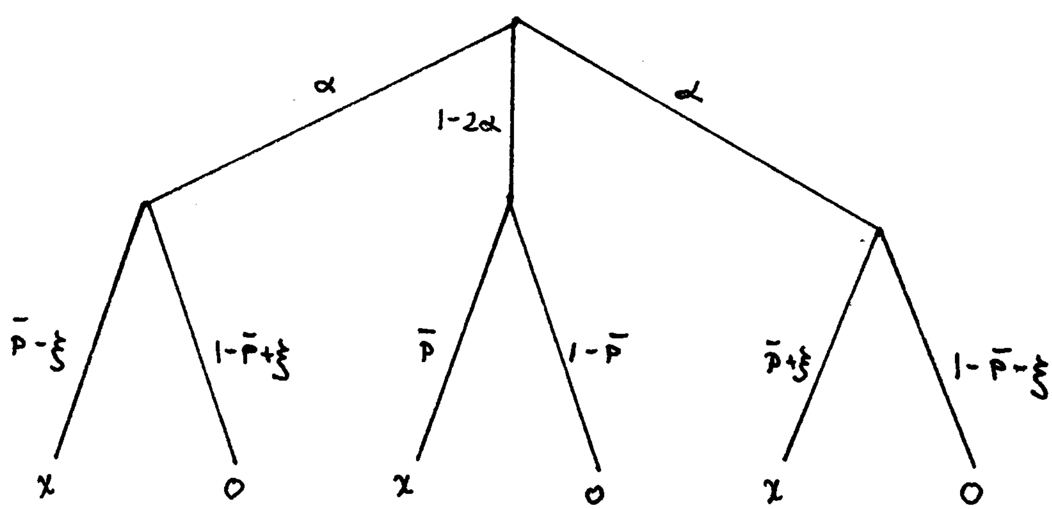

Figure 2

Let $x > 0$. For each p, let $y(p) = CE(x,p;0,1-p) = u^{-1}(u(x)f(p))$. For every $0 < y < x$, let $p(y)$ be defined implicitly by $(x,p(y);0,1-p(y)) \sim (y,1)$ and explicitly by $p(y) = f^{-1}(\frac{u(y)}{u(x)})$. Let F^* be a distribution function over the possible values of the probability of S in the lottery $(x,S;0,\sim S)$. If the decision maker considers this ambiguous lottery as a two-stage lottery, then the probability that the certainty-equivalence of $(x,S;0,\sim S)$ is not greater than y is given by $G(y) = F^*(f^{-1}(\frac{u(y)}{u(x)}))$. Let α^* and β^* be the minimal and the maximal possible values of the probability of S, that is, $\alpha^* = \sup\{p: F^*(p) = 0\}$ and $\beta^* = \inf\{p: F^*(p) = 1\}$. By (2.3), the value of the ambiguous lottery $(x,S;0,\sim S)$ equals

$$u(x)f(\alpha^*) + \int_{u^{-1}[u(x)f(\alpha^*)]}^{u^{-1}[u(x)f(\beta^*)]} u'(y)f(1-G(y))dy =$$

$$u(x)f(\alpha^*) + \int_{u^{-1}[u(x)f(\alpha^*)]}^{u^{-1}[u(x)f(\beta^*)]} u'(y)f(1-F^*(f^{-1}(\frac{u(y)}{u(x)})))dy \tag{3.1}$$

Substitute $z = f^{-1}(\frac{u(y)}{u(x)})$ into (3.1) and obtain

$$u(x)f(\alpha^*) + u(x)\int_{\alpha^*}^{\beta^*} f'(z)f(1-F^*(z))dz \tag{3.2}$$

Integrating (3.2) by parts yields

$$u(x)\int_{\alpha^*}^{\beta^*} f(z)f'(1-F^*(z))F^{*\prime}(z)dz \tag{3.3}$$

If the range of F^* is finite such that the possible probabilities of S are $\alpha^* = p_1 < \ldots < p_m = \beta^*$, then by (2.3) the value of the ambiguous lottery $(x,S;0,\sim S)$ is

$$u(x)f(p_1) + u(x)\sum_{i=2}^{m}[f(p_i) - f(p_{i-1})]f(\sum_{j=i}^{m} P^*(p_j)) \tag{3.4}$$

If $f(z) = z$, that is, if Anticipated Utility Theory reduces to Expected Utility Theory, then

$$u(x)\int_{\alpha^*}^{\beta^*} f(z)f'(1-F^*(z))F^{*\prime}(z)dz =$$

$$u(x)\int_{\alpha^*}^{\beta^*} zF^{*\prime}(z)dz = \bar{p}u(x)$$

where $\bar{p}$ is the expected value of the distribution function F^*. In this case, $(x,\bar{p};0,1-\bar{p})$ and $(x,S;0,\sim S)$ are equally attractive.

Consider now the case where $x < 0$. Let $\bar{f}(p) = 1 - f(1-p)$. By (2.2) the value of the lottery $(x,p;0,1-p)$ is $u(x)\bar{f}(p)$ (as before, $u(0) = 0$). In the same way as before, it can be proved that the value of the ambiguous lottery $(x,S;0,\sim S)$ equals

$$u(x)\bar{f}(\beta^*) + \int_{u^{-1}[u(x)\bar{f}(\beta^*)]}^{u^{-1}[u(x)\bar{f}(\alpha^*)]} u'(y)f(F^*(\bar{f}^{-1}(\frac{u(y)}{u(x)})))dy =$$

$$u(x)\bar{f}(\alpha^*) + u(x) \int_{\alpha^*}^{\beta^*} \bar{f}'(z)\bar{f}(1-F^*(z))dz = \tag{3.5}$$

$$u(x) \int_{\alpha^*}^{\beta^*} \bar{f}(z)\bar{f}'(1-F^*(z))F^{*\prime}(z)dz \tag{3.6}$$

If $x < 0$ and F^* is discrete, then (3.5) reduces to

$$u(x)\bar{f}(p_1) + u(x) \sum_{i=2}^{m} [\bar{f}(p_i) - \bar{f}(p_{i-1})]\bar{f}(\sum_{j=i}^{m} P^*(p_j)). \tag{3.7}$$

4. Ambiguous vs. Nonambiguous Lotteries

In this section I present the main result of this paper, that under several conditions, a nonambiguous lottery is preferred to an ambiguous one. Let F^* be a distribution function over the possible values of the probability of S in the lottery $(x,S;0,\sim S)$. Denote the mean of F^* by $\bar{p}$. Consider first the situation where the decision maker's beliefs, as represented by F^*, are symmetric around $\bar{p}$. That is, for every ξ, $F^*(\bar{p}+\xi) + F^*(\bar{p}-\xi) = 1$. In such a case, the decision maker believes that the probability of S being higher than $\bar{p} + \xi$ is equally probable to its being less than $\bar{p} - \xi$. This assumption looks plausible in the Ellsberg paradox (Problems 1 and 2). In Problem 1, for example, the decision maker does not know the number of red and black balls in the urn, hence the combination of i red balls and 100 - i black balls should seem as probable to him as the combination of 100 - i red balls and i black balls.

Theorem 4.2 presents conditions under which a nonambiguous lottery is preferred to an ambiguous one. For the proof of this theorem I need the following definition and Lemma.

Definition: The elasticity of a (differentiable) function $g: \mathbb{R} \to \mathbb{R}$ is given by $\frac{xg'(x)}{g(x)}$.

Lemma 4.1: The elasticity of the anticipated utility decision-weights function f is nondecreasing iff $f(p)f(q) \leq f(pq)$.

Proof: All proofs appear in the Appendix.

Theorem 4.2: Let F^* be a distribution function over the possible values of the probability of S in the ambiguous lottery $(x,S;0,\sim S)$. Assume that F^* is symmetric around $\bar{p}$. If f is convex, if its elasticity is nondecreasing, and if the elasticity of $\bar{f}$ is nonincreasing, then $(x,\bar{p};0,1-\bar{p})$ is preferred to $(x,S;0,\sim S)$.

Remarks: Increasing elasticity of f does not imply decreasing elasticity of $\bar{f}$. For example, the elasticity of $f(p) = 1 - \sqrt{1-p}$ is increasing while the elasticity of $\bar{f}(p) = \sqrt{p}$ is constant. $f(p) = \frac{e^p-1}{e-1}$ and $f(p) = p^t$, $t > 1$, satisfy all the conditions of Theorem 4.2. For the connection between the elasticity of f and the common ratio effect, see Segal (1984). For discussion of the convexity of f, see Yaari (1984, 1985a), Chew, Karni, and Safra (1985), and Segal (1984). (See also Section 5 below.)

Theorem 4.2 does not hold for every convex function f. Let f be as follows:

$$f(p) = \begin{cases} 0.99p & p < 0.999 \\ 10.99p - 9.99 & p > 0.999 \end{cases}$$

and let $P^*(1) = P^*(0.998) = \frac{1}{2}$. In this case, $\bar{p} = 0.999$ and $V(1,S;0,\sim S) = 0.994u(1) > 0.989u(1) = V(1,0.999;0,0.001)$. f is not a strictly convex function, but one can easily find a strictly convex function f based on this example for which Theorem 4.2 does not hold.

Consider now the case where F^* is nonsymmetric.

Theorem 4.3: Let F^* be a distribution function over the possible values of the probability of S in the ambiguous lottery $(x,S;0,\sim S)$, denote its mean value by $\bar{p}$, and let f be convex and twice differentiable.

(a) $x > 0$: If $\frac{f''}{f'}$ is nonincreasing and if the elasticity of f is nondecreasing, then $(x,\bar{p};0,1-\bar{p})$ is preferred to $(x,S;0,\sim S)$.

(b) $x < 0$: If $\frac{\bar{f}''}{\bar{f}'}$ is nondecreasing and if the elasticity of $\bar{f}$ is nonincreasing, then $(x,\bar{p};0,1-\bar{p})$ is preferred to $(x,S;0,\sim S)$.

Remarks: Nondecreasing elasticity does not imply that $\frac{f''}{f'}$ is nonincreasing. Let $f(p) = p^2$, $g(p) = \frac{e^p-1}{e-1}$, $h(p) = \frac{p+p^{10}}{2}$. These are convex functions with nondecreasing elasticities, but $\frac{f''}{f}$ is decreasing, $\frac{g''}{g'}$ is constant, and $\frac{h''}{h'}$ is first increasing, then decreasing.

Theorem 4.3-a does not hold for every convex function f such that its elasticity is nondecreasing. Let $f(p) = \frac{p+p^{10}}{2}$ and let $P^*(0.49) = \frac{50}{51}$, $P^*(1) = \frac{1}{51}$. In this case, $\bar{p} = 0.5$ and $V(1,S;0,\sim S) = 0.2528u(1) >$ $0.2505u(1) = V(1,0.5;0,0.5)$.

$\frac{f''}{f'}$ is decreasing iff $\frac{\bar{f}''}{\bar{f}'}$ is decreasing. The only functions f for which $\frac{f''}{f'}$ is nonincreasing and $\frac{\bar{f}''}{\bar{f}'}$ is nondecreasing are $f(p) = \frac{e^{\alpha p}-1}{e^{\alpha}-1}$. Recall, however, that the conditions of Theorem 4.3 are sufficient, but not necessary ones. One may be ambiguity averse even if his decision-weights function f does not satisfy all the conditions of Theorem 4.3. For $\frac{f''}{f'}$ as a measure of risk aversion, see Yaari (**1985a**) and Chew, Karni, and Safra (**1985**).

For certain distribution functions and for certain decision-weights functions (which do not satisfy the conditions of Theorem 4.3), one may be ambiguity averse for $x > 0$ and an ambiguity lover for $x < 0$. Let $f(x) = 1 - \cos(\frac{\pi x}{2})$, $P^*(0.2) = 0.25$, $P^*(1) = 0.75$, $\bar{p} = 0.8$. $V(1,S;0,\sim S) = 0.636u(1) < 0.691u(1) = V(1,0.8;0,0.2)$. On the other hand, $V(-1,S;0,\sim S) = 0.947u(-1) > 0.951u(-1) = V(-1,0.8;0,0.2)$.

Ellsberg suggested that in Problems 1 and 2 most people prefer the clear lotteries to the ambiguous ones. However, there exist situations where most people prefer the ambiguous options. Consider the following problem.

<u>Problem 3</u>: Urn I contains 1000 balls numbered from 1 to 1000. Urn II contains 1000 balls labeled by integers between 1 and 1000 (inclusive), but you do not know the exact composition of this urn. At lottery i, i = I,II, one ball is to be drawn at random out of urn i. If its number lies in the set $N \quad \{1,\ldots,1000\}$, the player receives $1000. Ellsberg predicted that if $\#N$ is small enough, most people will prefer II to I. (See Becker and Brownson (**1964**, footnote 4).)

I now present an example consistent with all the predictions of Problems 1-3 above. Let

$$f(p) = \begin{cases} 0.1p & p < 0.001 \\ \frac{1111p}{1110} - \frac{1}{1110} & p > 0.001 \end{cases}$$

Problem 1: The decision maker's beliefs concerning urn I are given by $P^*(25R) = P^*(75R) = 0.5$. $V(R_I) = V(B_I) = 0.49932u(100) < 0.49955u(100) = V(R_{II}) = V(B_{II})$.[2]

Problem 2: The decision maker's beliefs concerning this urn are given by $P^*(20B) = P^*(40B) = 0.5$. $V(A_1) = 0.33273u(100) > 0.33263u(100) = V(A_2)$. $V(A_3) = 0.66627u(100) < 0.66637u(100) = V(A_4)$.

Problem 3: Let $\#N = 1$. The decision maker's beliefs concerning this urn are given by $P^*(0) = 0.9$, $P^*(0.01) = 0.1$, $\bar{p} = 0.001$. $V(I) = 0.0001u(1000) < 0.0009u(1000) = V(II)$.

Ambiguous lotteries and two-stage lotteries were compared in experiments done by Yates and Zakowski (1976). Consider the following problems. (In all cases, the decision maker decides upon his color first.)

G : An urn contains 5 blue balls and 5 red balls, one ball is drawn at random. If its color matches that chosen, the player receives $1.

G': An urn contains 11 balls, numbered from 0 to 10. One ball is drawn at random. If its number is i, then i blue balls and $10-i$ red balls are put in a second urn from which one ball is drawn at random. If its color matches that chosen, the player receives $1.

G": An urn contains 10 balls, blue and red, but of unknown composition. One ball is drawn at random. If its color matches that chosen, the player receives $1.

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[2] The numbers are relatively close. However, the aim of this example is to prove the existence of such a function, and not to describe real preferences.

78% of the subjects preferred G to G'' and 68% preferred G' to G''.[3] The first preference agrees with Theorem 4.2. It also seems reasonable to assume that the number of subjects who prefer G to G'' exceeds the number of those who prefer G' to G''. The following example shows G' preferred to G''.

The distribution function in G' is given by $P^*(\frac{i}{10}) = \frac{1}{11}$, $i = 0,\ldots, 10$. Let Q^* be the decision maker's distribution function in G'', where $Q^*(0) = Q^*(1) = \frac{1}{22}$, $Q^*(\frac{1}{10}) = Q^*(\frac{9}{10}) = \frac{3}{22}$, $Q^*(\frac{2}{10}) = \ldots = Q^*(\frac{8}{10}) = \frac{1}{11}$. Let $f(p) = p^2$. The value of G' is $0.16818u(1)$, while the value of G'' is only $0.16785u(1)$.

5. The Ellsberg Paradox and Other Non-Expected Utility Phenomena

In this section I discuss the connection between the Ellsberg paradox and the concept of risk aversion, as well as its connection to some other paradoxes in Expected Utility Theory.

Risk aversion suggests that decision makers always prefer a certain prize x_o to a random variable $\tilde{x}$ with expected value x_o. It is well known that in the Expected Utility framework, this condition is equivalent to the assumption that the utility function u is concave. Machina (1982) extended this concept for his functional, and Chew, Karni, and Safra (1985) recently gave similar conditions for Anticipated Utility Theory. According to their work, the Anticipated Utility functional (2.3)-(2.5) represents risk aversion behavior if and only if u is concave and f convex (see also Yaari (1985a)).

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[3] 53% preferred G to G'. Other experiments show, however, that decision makers prefer a one-stage lottery to a two-stage lottery with the same winning probabilities (see Kahneman and Tversky (1979)).

The connection between the convexity of f and risk aversion becomes apparent when applying Anticipated Utility Theory to the Allais paradox (Allais (1953)). Consider the following four lotteries.

$A_1 = (0,0.89;1000000,0.11)$

$A_2 = (0,0.9;5000000,0.1)$

$B_1 = (1000000,1)$

$B_2 = (0,0.01;1000000,0.89;5000000,0.1)$

Allais found that most people prefer A_2 to A_1 but B_1 to B_2, although by Expected Utility Theory $A_2 \succsim A_1$ iff $B_2 \succsim B_1$. Similar results were reported by Kahneman and Tversky (1979) and by MacCrimmon and Larsson (1979) who repeated this experiment for various values of prizes and probabilities. The general form of the Allais paradox involves the following lotteries:

$A_1^* = (0,1-p;x,p)$

$A_2^* = (0,1-q;y,q)$

$B_1^* = (0,1-p-r;x,p+r)$

$B_2^* = (0,1-q-r;x,r;y,q)$

where $0 < x < y$, $p > q$. By Expected Utility Theory, $A_2^* \succsim A_1^*$ iff $B_2^* \succsim B_1^*$, while empirical evidence shows that if $A_2^* \sim A_1^*$, then $B_1^* \succ B_2^*$. Segal (1985) proved that such behavior is consistent with Anticipated Utility Theory iff f is convex. Similarly, the probabilistic insurance phenomenon (Kahneman and Tversky (1979)) is resolved within the Anticipated Utility framework given that f is convex (Segal (1986)).

In the Ellsberg urn there is only one possible prize, hence the shape of the utility function is not relevant for this discussion, but the importance of the convexity of f becomes clear from Theorems 4.2 and 4.3. To illustrate this, consider Problem 1 of the Introduction, and assume $P^*(\frac{1}{4}) = P^*(\frac{3}{4}) = \frac{1}{2}$. By (3.4), the value of the ambiguous lottery $(x,S;0,\sim S)$ is

$f(\frac{1}{4}) + [f(\frac{3}{4}) - f(\frac{1}{4})]f(\frac{1}{2}) = f(\frac{1}{4})[1 - f(\frac{1}{2})] + f(\frac{3}{4})f(\frac{1}{2})$, while $V(x,\frac{1}{2};0,\frac{1}{2}) = f(\frac{1}{2})$. Let $f(\frac{1}{2}) = \alpha$. If f is concave, then $f(\frac{1}{4}) > \frac{\alpha}{2}$, and $f(\frac{3}{4}) > \frac{1+\alpha}{2}$, hence $V(x,S;0,\sim S) > \alpha = V(x,\frac{1}{2};0,\frac{1}{2})$, in contradiction to the common attitude towards this problem.

The second condition of Theorem 4.2, concerning the elasticity of f, has direct relevance to another phenomenon, called the common ratio effect (MacCrimmon and Larsson (1979)). Consider the following lotteries.

$$C_1(p) = (0,1-p;x,p)$$

$$C_2(p) = (0,1-0.8p;5x,0.8p).$$

By Expected Utility Theory, $C_1(p) \succsim C_2(p) \Leftrightarrow C_1(q) \succsim C_2(q)$, but MacCrimmon and Larsson found that if $C_1(p) \sim C_2(p)$ and $q > p$, then $C_1(q) \succ C_2(q)$. Segal (1985) proved that such behavior is consistent with Anticipated Utility Theory if and only if the elasticity of f is increasing. Thus, if an Anticipated Utility maximizer behaves according to the Allais paradox and the common ratio effect, then he will usually also reject the Ellsberg urn.

6. Increasing Ambiguity

In Section 4, I compared some ambiguity with none. In this section I discuss degrees of ambiguity. The function F^* reprsents the decision maker's ambiguity concerning the probability of the event S in the lottery $(x,S;0,\sim S)$. Therefore, in order to rank degrees of ambiguity, one should define an order on the set of the distribution functions F^*. For example, Rothschild and Stiglitz (1970) defined the order "more variable than" between distribution functions, while Jones and Ostroy (1984), in addition to creating their own definition, gave a justification for a restricted version of this order, called "star-shaped spreading of beliefs," as follows.

<u>Definition</u>: Let F^* and G^* be two distribution functions on $[0,1]$ and let $T(y) = \int_0^y [G^*(x) - F^*(x)]dx$. G^* is more variable than F^* $(G^* >_I F^*)$ iff $T(1) = 0$ and for every $0 < y < 1$, $T(y) > 0$.

<u>Definition</u>: Let F^* and G^* be two distribution functions on $[0,1]$. G^* is more ambiguous than F^* $(G^* >_A F^*)$ iff G^* is a star-shaped spreading of F^*, that is,

(a) F^* and G^* have the same mean value $\bar{p}$;

(b) $G^*(p) > F^*(p)$ for $p < \bar{p}$ and $G^*(p) < F(p)$ for $p > \bar{p}$.

Both $>_I$ and $>_A$ are transitive but not complete. Obviously, $>_A$ $>_I$, but $>_I$ $>_A$. Note that $T(1) = 0$ implies that F^* and G^* have the same mean value (see Rothschild and Stiglitz (1970)).

As proved in Theorem 4.2, a clear lottery is preferred to an ambiguous one. One is thus tempted to assume that if G^* is more ambiguous than F^*, then the value of the lottery $(x,S;0,\sim S)$ under F^* is greater than its value under G^*. This supposition does not generally hold true, as demonstrated by the following example.

Let $P^*_{\xi,\alpha}(\bar{p}-\xi) = P^*_{\xi,\alpha}(\bar{p}+\xi) = \alpha$, $P^*_{\xi,\alpha}(\bar{p}) = 1 - 2\alpha$. Obviously, $P^*_{\xi',\alpha} >_A P^*_{\xi,\alpha}$ iff $\xi' > \xi$. Let $x > 0$ and assume without loss of generality that $u(x) = 1$. The value of the lottery $(x,S;0,\sim S)$ under $P^*_{\xi,\alpha}$ is $h(\xi,\alpha) = f(\bar{p}-\xi) + [f(\bar{p}) - f(\bar{p}-\xi)]f(1-\alpha) + [f(\bar{p}+\xi) - f(\bar{p})]f(\alpha)$. Differentiating h with respect to ξ yields

$$\frac{\partial}{\partial\xi} h(\xi,\alpha) = -[1 - f(1-\alpha)]f'(\bar{p}-\xi) + f(\alpha)f'(\bar{p}+\xi).$$

If f is convex, then $\frac{\partial}{\partial\xi} h(0,\alpha) < 0$, but for $\bar{p} = \alpha = \frac{1}{2}$, $\frac{\partial}{\partial\xi} h(\frac{1}{2},\frac{1}{2}) > 0$. Similarly, $P^*_{\xi,\alpha'} >_A P^*_{\xi,\alpha}$ iff $\alpha' > \alpha$. Differentiating h with respect to α yields

$$\frac{\partial}{\partial\alpha} h(\xi,\alpha) = -[f(\bar{p}) - f(\bar{p}-\xi)]f'(1-\alpha) + [f(\bar{p}+\xi) - f(\bar{p})]f'(\alpha)$$

If f is convex, then $\frac{\partial}{\partial\alpha}(\xi,0) < 0$, but $\frac{\partial}{\partial\alpha}(\xi,\frac{1}{2}) > 0$.

Under certain conditions $G^* >_A F^*$ implies that the value of $(x,S;0,\sim S)$ under G^* is not greater than its value under F^*.

Definition: Let $0 < p < 1$. H^*_p is the uniform distribution on $[0,2p]$ when $p < \frac{1}{2}$, and the uniform distribution on $[2p-1,1]$ when $p > \frac{1}{2}$.

Theorem 6.1: Let G^* and F^* be symmetric around $\bar{p}$ such that $H^*_{\bar{p}} >_A G^* >_A F^*$. If f is convex, if the elasticity of f' is nondecreasing, and if the elasticity of $\bar{f}'$ is nonincreasing, then the value of the ambiguous lottery $(x,S;0,\sim S)$ under F^* is greater than its value under G^*.

Remarks: $f(p) = \frac{e^p-1}{e-1}$ and $f(p) = p^t$, $t > 1$, satisfy all three conditions of Theorem 5.1.

If the density function of G^* is a single-peak function, i.e., $G^{*\prime\prime}(p) > 0$ for $p < \bar{p}$ and $G^{*\prime\prime}(p) < 0$ for $p > \bar{p}$, then $H^*_{\bar{p}} >_A G^*$.

The elasticity of f', $\frac{pf''}{f'}$, is similar to Arrow's measure of relative risk aversion (Arrow (1974)).

Theorem 6.1 does not hold for $>_I$, even with symmetric distribution functions (Example 1 below), nor does it hold for $>_A$ with nonsymmetric distribution functions (Example 2 below).

Example 1: Let $P^*(\frac{1}{4}) = P^*(\frac{3}{4}) = \frac{1}{4}$, $P^*(\frac{1}{2}) = \frac{1}{2}$; let $Q^*(\frac{1}{32}) = Q^*(\frac{1}{16}) = Q^*(\frac{11}{32}) = Q^*(\frac{21}{32}) = Q^*(\frac{15}{16}) = Q^*(\frac{31}{32}) = \frac{1}{32}$, $Q^*(\frac{5}{16}) = Q^*(\frac{11}{16}) = \frac{5}{32}$, $Q^*(\frac{1}{2}) = \frac{1}{2}$; and let $f(p) = p^2$. The mean value of P^* and Q^* is $\frac{1}{2}$, $H^*_{0.5} >_A Q^* >_I P^*$, but not $Q^* >_A P^*$. The value of the ambiguous lottery $(1,S;0,\sim S)$ under P^* is $0.1875u(1)$, while its value under Q^* is $0.18774u(1)$.

Example 2: Let $P^*(\frac{1}{4}) = P^*(\frac{3}{4}) = \frac{1}{4}$, $P^*(\frac{1}{2}) = \frac{1}{2}$; let $Q^*(\frac{1}{16}) = Q^*(\frac{1}{8}) = Q^*(\frac{5}{8}) = Q^*(\frac{11}{16}) = \frac{1}{16}$, $Q^*(\frac{1}{4}) = \frac{1}{8}$, $Q^*(\frac{1}{2}) = \frac{3}{8}$, $Q^*(\frac{3}{4}) = \frac{1}{4}$; and let $f(p) = p^2$. The mean value of P^* and Q^* is $\frac{1}{2}$, and $H^*_{0.5} >_A Q^* >_A P^*$. The value of the ambiguous lottery $(1,S;0,\sim S)$ under P^* is $0.1875u(1)$, while its value under Q^* is $0.18896u(1)$.

7. Changes in $\bar{p}$

Consider the ambiguous lottery $(x,S;0,\sim S)$, $x < 0$. The decision maker does not know the exact probability of S, but he believes that it lies between α^* and β^* with a mean value of $\bar{p}$. Let $y < 0$ be the insurance premium he is willing to pay to insure himself against S. That is, $(y,1) \sim (x,S;0,\sim S)$. Suppose that the decision maker's beliefs change and he now believes that the probability of S is greater than he originally thought. For example, suppose that he discovers that his car is not as safe as he thought it to be, thus it is more likely to involve him in an accident. He remains unsure of the probability of S (getting into an accident, in this example), but there is a rightward shift in his beliefs on this probability, represented by the distribution function F^*. Obviously, he is now willing to pay more to insure himself against S. It is not clear, however, what will happen to the ratio between y and the expected value of the lottery $(x,S;0,\sim S)$, $y/\bar{p}x$. This question has been investigated by Hogarth and Kunreuther (1984) by means of laboratory experiments. They found that when $x < 0$, $y/\bar{p}x$ is decreasing with $\bar{p}$, that is, as $\bar{p}$ increases, decision makers are willing to pay relatively less to insure themselves against S. This prediction agrees with some real world observations. For example, the weekly insurance rate for rented cars exceeds $\frac{1}{52}$ of an equivalent regular, annual insurance premium, even taking into account that a rented car will likely have a higher weekly mileage than a privately owned car. Similarly, short-term health insurance is relatively more expensive than long term insurance (although these may be the result of price discrimination in the insurance market). Similar results appear in this section.

For the sake of simplicity, assume that u is the identity function, i.e., $u(x) = x$. For a discussion of Anticipated Utility Theory with linear utility function, see Yaari (**1984**).

In previous sections, the function F^* represented the decision maker's beliefs on the possible values of the probability of S, but in this section, the function F^* changes with $\bar{p}$. A discussion of two such possible changes, a parallel change and a proportional one, follows.

<u>Definition</u>: G^* is obtained from F^* by a positive parallel change if there exists $\varepsilon > 0$ such that for every p $G^*(p+\varepsilon) = F^*(p)$.

<u>Definition</u>: G^* is obtained from F^* by a proportional change if there exists $\delta > 1$ such that $G^*(\delta p) = F^*(p)$.

Let F^* be symmetric on $[0,2\xi^*]$ and define, for $\varepsilon > 0$, $G^*_\varepsilon(p) = F^*(p-\varepsilon)$. Obviously, G^*_ε is obtained from F^* by a positive parallel change, and its mean value is $\bar{p} = \xi^* + \varepsilon$. Let $y = CE(x,S;0,\sim S)$. That is, $y = x \int_{\bar{p}-\xi^*}^{\bar{p}+\xi^*} f(z)f'(1-F^*(z-\bar{p}+\xi^*))F^{*\prime}(z-\bar{p}+\xi^*)dz$ for $x > 0$, and the same with $\bar{f}$ instead of f for $x < 0$ (see (3.3) and (3.6)).

<u>Theorem 7.1</u>: Let F^* be symmetric on $[0,2\xi^*]$ such that $H_{\xi^*} >_A F^*$. If f is convex, if the elasticity of f' is nondecreasing, and if the elasticity of $\bar{f}'$ is nonincreasing, then $\frac{\partial}{\partial\varepsilon}\frac{y}{\bar{p}x} > 0$ for $x > 0$ and $\frac{\partial}{\partial\varepsilon}\frac{y}{\bar{p}x} < 0$ for $x < 0$.

Remark: Theorem 7.1 does not hold true for every F^*. Let $f(p) = p^{1.5}$ and let $P^*(0) = P^*(0.99) = 0.5$. For $\bar{p} = 0.495$, $\frac{y}{\bar{p}x} = 0.7036$, while for $\bar{p} = 0.505$, $\frac{y}{\bar{p}x} = 0.7014$.

I now discuss proportional changes. Let $\bar{p}$ be the mean value of F^* and it follows that the mean value of G^* is $\delta\bar{p}$. Let α^* and β^* be the minimal and the maximal possible values of the probability of S according to F^*, and let $x > 0$. By (3.2), the value of the ambiguous lottery $(x,S;0,\sim S)$ under G^* equals

$$xf(\delta\alpha^*) + x\int_{\delta\alpha^*}^{\delta\beta^*} f'(z)f(1-F^*(\tfrac{z}{\delta}))dz =$$

$$xf(\delta\alpha^*) + \delta x\int_{\alpha^*}^{\beta^*} f'(\delta z)f(1-F^*(z))dz$$

If the decision maker is indifferent to the choice between $(x,S;0,\sim S)$ and $(y,1)$, then the ratio between y and the expected value of $(x,S;0,\sim S)$ under G^* is given by

$$\frac{y}{\delta\bar{p}x} = \frac{f(\delta\alpha^*)}{\delta\bar{p}} + \frac{1}{\bar{p}}\int_{\alpha^*}^{\beta^*} f'(\delta z)f(1-F^*(z))dz = \psi(\delta)$$

Let $x < 0$. In this case,

$$\frac{y}{\delta\bar{p}x} = \frac{\bar{f}(\delta\alpha^*)}{\delta\bar{p}} + \frac{1}{\bar{p}}\int_{\alpha^*}^{\beta^*} \bar{f}'(\delta z)\bar{f}(1-F^*(z))dz = \bar{\psi}(\delta)$$

Theorem 7.2: If f is convex, then $\psi'(\delta) > 0$ and $\bar{\psi}'(\delta) < 0$.

8. Interpersonal Comparisons

Connection between Ellsberg paradox and risk aversion is very intuitive. As ambiguous lotteries appear to be riskier than clear lotteries, it is natural to expect that a decision maker with a higher degree of risk aversion is willing to pay less to participate in a certain ambiguous lottery. The Arrow-Pratt measure of risk aversion is defined for the Expected Utility functional, but the concept of risk aversion does not depend on this theory. Yaari (1985a) suggests two possible measurements of risk aversion through the decision-weights function f, one of which proves itself to be useful in comparing decision makers' willingness to pay for a certain ambiguous lottery.

Let $(x,S;0,\sim S)$ be an ambiguous lottery. Let two decision makers have the same beliefs and the same decision-weights function f, but assume that they differ in their utility functions, denoted by u and v. $u(0) = v(0) = 0$ and, without loss of generality, $u(x) = v(x)$. By definition, each decision maker is willing to pay its certainty equivalence for a lottery (Section 2). Since both decision makers give the lottery $(x,S;0,\sim S)$ the same value, denote it by α, it follows that the first decision maker is willing to pay more than the second one iff $u^{-1}(\alpha) > v^{-1}(\alpha)$.

More interesting is the comparison of two decision makers who differ only in their decision-weights functions. Let I and II have the same utility function u and the same distribution function F^*, and denote their decision-weights functions by f and g respectively.

<u>Theorem 8.1</u>: Let $x > 0$. If f and g are convex, and if for every p, $f(p) > g(p)$, then I is willing to pay more for the lottery $(x,S;0,\sim S)$ than II is willing to pay. If $x < 0$, I is willing to pay less to insure himself against S than II is willing to pay.

<u>Remark</u>: Yaari (1985a) suggested two (different) definitions of one decision maker (with g) as more risk averse than a second decision maker (with f):

R1. $g = h \circ f$ where h is convex

R2. $g(p) \leq f(p)$ for all p.

As pointed out by Yaari, R1 implies R2, but R2 does not imply R1. For example, let $f(p) = 0.25p$ for $p < 0.5$ and $1.5p - 0.5$ for $p > 0.5$, and let $g(p) = p^2$. For every p, $g(p) < f(p)$, but there is no convex function h such that $g = h \circ f$. Indeed, if h is convex and if $h(p) = p$ for $p = 0, 0.25, 1$, then $h(p) = p$ for all p. Theorem 8.1 suggests a justification for Yaari's second definition.

9. Some Remarks on the Literature

Problem 1 and Problem 2 of the Introduction were presented by Ellsberg in 1961 to support the claim that risk and uncertainty are two different concepts, and to prove that decision makers do not always accept Savage's Sure Thing Principle. Ellsberg himself did not conduct any empirical research, but his predictions were later confirmed by Becker and Brownson (1964), MacCrimmon (1965), Slovik and Tversky (1974), Yates and Zukowski (1976), and MacCrimmon and Larsson (1979). Besides empirical evidence, certain theoretical solutions to this paradox were suggested. For these, and for discussions of the relevance of the Ellsberg paradox, see Fellner (1961, 1963), Raiffa (1961), Brewer (1963), Ellsberg (1963), Roberts (1963), Brewer and Fellner (1965), Schneeweiss (1968), Smith (1969), Dreze (1974), Sherman (1974), Gardenfors and Sahlin (1982, 1983), Fishburn (1983), Schmeidler (1984), and Einhorn and Hogarth (1985). For review articles of these subjects, see MacCrimmon and Larsson (1979), Schoemaker (1982), and Machina (1983).

Three recent models try to explain the Ellsberg paradox by using nonadditive probabilities. Einhorn and Hogarth (1985) suggest that when presented with ambiguous situations, decision makers use the expected probability ($\frac{1}{2}$ in Problem 1, $\frac{1}{3}$ in Problem 2-A_2, etc.) as an anchor, but adjust this value according to the other possible values of the probability of the ambiguous event. An axiomatic basis for nonadditive probabilities was suggested by Schmeider (1984), replacing the IA by a weaker independence assumption.

Fishburn (1983) presented another model in which probabilities are not additive. He proved the existence of a function ρ such that A is more probable than B iff $\rho(A,B) > 0$ and used this function to explain the Ellsberg paradox. In this model, preferences are not assumed to be transitive, nor do they satisfy the IA. As Fishburn proved (Theorem 3), adding these two assumptions implies that $\rho(A,B) = P(A) - P(B)$, but in such a case ρ fails to explain the Ellsberg paradox.

These last three models all agree that probabilities are not necessarily additive. This paper tried a different approach, in which probabilities are additive, but not multiplicative. That is, preference relations satisfy the IA, but not the RCLA. Except for this axiom, this model satisfies all the usually accepted assumptions such as transitivity and First Order Stochastic Dominance.

The idea of modelling the Ellsberg urn as a two-stage lottery, although within the Expected Utility framework, was suggested by Gardenfors and Sahlin (1982, 1983). In their model, the decision maker's approach towards the Ellsberg urn consists of two stages. First, he decides what possible combinations of balls are unlikely to happen. Then he considers the urn as an Expected Utility maximizer, while assuming the worst out of the remaining possible combinations. Of course, this approach does not assume a real two-stage lottery, as an Expected Utility maximizer certainly does not consider a two-stage lottery by this maximin rule.

This paper discussed changes in the distribution function F^*. For other discussions of increasing risk and Anticipated Utility Theory, see Quiggin (1982), Yaari (1984, 1985a), Chew, Karni, and Safra (1985) and Segal (1984).

Appendix

Proof of Lemma 4.1

Let $r = pq$, $q = \frac{r}{p}$, and $\lambda(p) = f(p)f(\frac{r}{p})$. For $p=r$ and for $p=1$, $f(p)f(q) = f(pq)$. For $r < p < 1$,

$$\lambda'(p) \lesseqgtr 0 \iff \frac{f(p)f(\frac{r}{p})}{p} \left[\frac{pf'(p)}{f(p)} - \frac{\frac{r}{p}\, f'(\frac{r}{p})}{f(\frac{r}{p})}\right] \lesseqgtr 0.$$

If the elasticity of f is increasing, λ is decreasing on $(r,\sqrt{r})$ and increasing on $(\sqrt{r},1)$, and if the elasticity of f is decreasing, λ is increasing on $(r,\sqrt{r})$ and decreasing on $(\sqrt{r},1)$, hence the lemma.

Proof of Theorem 4.2

Since F^* is increasing and bounded, it is sufficient to prove the theorem only for those functions F^* which have a finite range.

Let $x > 0$ and let P^* be a symmetric probability function on $\bar{p} - \xi^* = \bar{p} - \xi_k, \ldots, \bar{p} - \xi_1, \bar{p}, \bar{p} + \xi_1, \ldots, \bar{p} + \xi_k = \bar{p} + \xi^*$. That is, for every i, $P^*(\bar{p}-\xi_i) = P^*(\bar{p}+\xi_i)$. Let $\xi_0 = 0$ and let $\alpha_i = \frac{1}{2} - P^*(p < \bar{p} - \xi_i)$, $i = 1,\ldots,k$. Since $u(x) > 0$, the theorem holds iff

$$(4.1) \quad \sum_{i=1}^{k} \{[f(\bar{p}-\xi_{i-1})-f(\bar{p}-\xi_i)]f(\tfrac{1}{2} + \alpha_i) + [f(\bar{p}+\xi_i)-f(\bar{p}+\xi_{i-1})]f(\tfrac{1}{2} - \alpha_i)\} < f(\bar{p}) - f(\bar{p}-\xi^*)$$

Let $g_i(\gamma) = [f(\bar{p}-\xi_{i-1})-f(\bar{p}-\xi_i)]f(\frac{1}{2} + \gamma) + [f(\bar{p}+\xi_i)-f(\bar{p}+\xi_{i-1})]f(\frac{1}{2} - \gamma)$.

$$(4.2) \qquad \frac{\partial g_i(\gamma)}{\partial \gamma} = [f(\bar{p}-\xi_{i-1}) - f(\bar{p}-\xi_i)]f'(\tfrac{1}{2} + \gamma) - [f(\bar{p}+\xi_i) - f(\bar{p}+\xi_{i-1})]f'(\tfrac{1}{2} - \gamma) \gtreqless 0 \iff$$

$$H(\gamma) = \frac{f'(\frac{1}{2} + \gamma)}{f'(\frac{1}{2} - \gamma)} \gtreqless \frac{f(\bar{p}+\xi_i) - f(\bar{p}+\xi_{i-1})}{f(\bar{p}-\xi_{i-1}) - f(\bar{p}-\xi_i)} = K_i$$

Since f is convex, it follows that $H(\gamma)$ is increasing, and $H(0) < K_i < H(\frac{1}{2})$. There exists therefore γ^* such that (4.2) holds iff $\gamma \gtreqless \gamma^*$. It thus follows that if $\alpha_i > \gamma^*$, then the left-hand side of (4.1) increases by setting $\tilde{P}^*(\bar{p}-\xi_i) = \tilde{P}^*(\bar{p}+\xi_i) = 0$, $\tilde{P}^*(\bar{p}-\xi_{i-1}) = \tilde{P}^*(\bar{p}+\xi_{i-1}) = P^*(\bar{p}-\xi_{i-1}) + P^*(\bar{p}-\xi_i)$, and if $\alpha_i < \gamma^*$, then the left-hand side of (4.1) increases by setting $\tilde{P}^*(\bar{p}-\xi_{i-1}) = \tilde{P}^*(\bar{p}+\xi_{i-1}) = 0$, $\tilde{P}^*(\bar{p}-\xi_i) = \tilde{P}^*(\bar{p}+\xi_i) = P^*(\bar{p}-\xi_{i-1}) + P^*(\bar{p}-\xi_i)$. By repeating this process k-1 times it follows that there exist α and $\xi < \xi^*$ such that the value of the ambiguous lottery $(x,S;0,\sim S)$ under P^{**} is greater than its value under P^*, where $P^{**}(\bar{p}-\xi) = P^{**}(\bar{p}+\xi) = \alpha$ and $P^{**}(\bar{p}) = 1 - 2\alpha$. It is sufficient therefore to prove that for every $0 < \alpha < \frac{1}{2}$,

$$[f(\bar{p}) - f(\bar{p}-\xi)]f(1-\alpha) + [f(\bar{p}+\xi) - f(\bar{p})]f(\alpha) < f(\bar{p}) - f(\bar{p}-\xi)$$

This inequality holds iff

$$\text{(4.3)} \qquad \frac{f(\bar{p}+\xi) - f(\bar{p})}{f(\bar{p}) - f(\bar{p}-\xi)} < \frac{1-f(1-\alpha)}{f(\alpha)}$$

Since f is convex, the right-hand side of (4.3) decreases with α, while the left-hand side increases with ξ. The maximal possible value of ξ is $\bar{p}$ when $\bar{p} < \frac{1}{2}$ and $1-\bar{p}$ when $\bar{p} > \frac{1}{2}$. Let $\bar{q} = 1 - \bar{p}$. (4.3) thus holds if $f(2\bar{p})/f(\bar{p}) < f(1)/f(\frac{1}{2})$ for $\bar{p} < \frac{1}{2}$ and if $\bar{f}(2\bar{q})/\bar{f}(\bar{q}) > \bar{f}(1)/\bar{f}(\frac{1}{2})$ for $\bar{p} > \frac{1}{2}$. By Lemma 4.1 these inequalities are satisfied if the elasticity of f is nondecreasing and if the elasticity of $\bar{f}$ is nonincreasing.

If $x < 0$, then the inequality sign in (4.1) is reversed. $\bar{f}$ is concave and in the same way as before it is sufficient to prove the theorem for P^{**}. Obviously (4.3) and

$$\frac{\bar{f}(\bar{p}+\xi) - \bar{f}(\bar{p})}{\bar{f}(\bar{p}) - \bar{f}(\bar{p}-\xi)} > \frac{1-\bar{f}(1-\alpha)}{\bar{f}(\alpha)}$$

are equivalent.

Proof of Theorem 4.3

As in the proof of Theorem 4.2, it is sufficient to prove the theorem only for those functions F^* which have a finite range. Let $x > 0$ and assume without loss of generality that $u(x) = 1$. Let $p_1 < \dots < p_m$ and consider the following constrained maximization problem:

$$\begin{aligned} &\text{maximize} && g(\alpha_1,\dots,\alpha_m) = f(p_1) + \sum_{i=2}^{m} [f(p_i) - f(p_{i-1})] f(\sum_{j=i}^{m} \alpha_j) \\ &\text{subject to} && \Sigma\alpha_i = 1 \\ &&& \Sigma\alpha_i p_i = \bar{p} \\ &&& \alpha_i \geq 0, \quad i = 1,\dots,m. \end{aligned}$$

Let $K = \sum_3^m \alpha_i$, $L = \sum_3^m \alpha_i p_i$. Solve α_1 and α_2 from the constraints, substitute into g, and obtain

$$\tilde{g}(\alpha_3,\dots,\alpha_m) = f(p_1) + [f(p_2) - f(p_1)] f(Kp_2+\bar{p}-p_1-L) + \sum_{i=3}^{m} [f(p_i) - f(p_{i-1})] f(\sum_{j=i}^{m} \alpha_j)$$

$$\frac{\partial^2 \tilde{g}}{\partial \alpha_3^2} = [f(p_2) - f(p_1)] f''(Kp_2+\bar{p}-p_1-L)(p_2-p_3)^2 + [f(p_3) - f(p_2)] f''(K) > 0$$

Hence, by the second order necessary conditions for maximum it follows that there is no inner solution to this maximization problem. By using this argument $m-2$ times it follows that all but at most two of $\alpha_1,\dots,\alpha_m$ are zero. It is therefore sufficient to prove that if $\alpha p_1 + (1-\alpha)p_2 = \bar{p}$, $p_1 < \bar{p} < p_2$, then

(4.4) $$f(p_1) + [f(p_2) - f(p_1)]f(1-\alpha) < f(\bar{p}).$$

Substitute into (4.4) $p_2 = \frac{\bar{p}-\alpha p_1}{1-\alpha}$ and obtain

(4.5) $$\pi(p_1) = f(p_1) + [f(\frac{\bar{p}-\alpha p_1}{1-\alpha}) - f(p_1)]f(1-\alpha) < f(\bar{p}).$$

f is convex, hence

$$\pi''(p_1) = f''(p_1)[1-f(1-\alpha)] + f''(\frac{\bar{p}-\alpha p_1}{1-\alpha}) \frac{\alpha^2}{(1-\alpha)^2} f(1-p) > 0$$

Inequality (4.5) is satisfied for $p_1 = \bar{p}$. It is therefore sufficient to prove it for $p_1 = 0$ when $\bar{p} < 1-\alpha$ and for $p_2 = 1$ when $\bar{p} > 1-\alpha$. By Lemma 4.1, if the elasticity of f is increasing, then $f(\frac{\bar{p}}{1-\alpha})f(1-\alpha) < f(\bar{p})$, hence (4.5) is satisfied for $p_1 = 0$.

Let $p_2 = 1$. Substitute into (4.4) $\bar{p} = \alpha p_1 + 1 - \alpha$ and obtain

$$\rho(p_1) = f(p_1) + [1-f(p_1)]f(1-\alpha) - f(\alpha p_1+1-\alpha) < 0.$$

$\rho(0) = \rho(1) = 0$, hence it is sufficient to prove that ρ does not have local maxima on $(0,1)$:

$$\rho'(p_1) = 0 \Rightarrow 1 - f(1-\alpha) = \frac{\alpha f'(\alpha p_1+1-\alpha)}{f'(p_1)}$$

At these critical points, $\rho''(p_1) > 0$ iff

(4.6) $$\frac{f''(p_1)}{f'(p_1)} > \frac{\alpha f''(\alpha p_1+1-\alpha)}{f'(\alpha p_1+1-\alpha)} .$$

$p_1 < \alpha p_1 + 1 - \alpha$, and (4.6) follows from the assumptions that f is convex and that $\frac{f''}{f'}$ is nonincreasing.

If $x < 0$, the proof is similar.

Proof of Theorem 6.1

Obviously, one can replace α^* and β^* in (3.2) and (3.5) by $\alpha < \alpha^*$ and $\beta > \beta^*$. Let $\xi^* = \inf\{p: G^*(p) = 0\}$ and let $x > 0$. The value of the ambiguous lottery $(x,S;0,\sim S)$ under F^* is not less than its value under G^* iff

$$\int_{\bar{p}-\xi^*}^{\bar{p}+\xi^*} f'(z)[f(1-F^*(z)) - f(1-G^*(z))]dz \geq 0$$

F^* and G^* are symmetric around $\bar{p}$, hence it is sufficient to prove that for all $0 < \xi < \xi^*$,

$$(6.1) \qquad \frac{f'(\bar{p}+\xi)}{f'(\bar{p}-\xi)} \leq \frac{f(1-F^*(\bar{p}-\xi)) - f(1-G^*(\bar{p}-\xi))}{f(1-G^*(\bar{p}+\xi)) - f(1-F^*(\bar{p}+\xi))}$$

Assume first that $\bar{p} \leq \frac{1}{2}$. $H^*_{\bar{p}} \geq_A G^*$, hence, on $[\bar{p}-\xi^*,\bar{p}]$ $F^*(p) \leq G^*(p) \leq \frac{p}{2\bar{p}}$, and on $[\bar{p},\bar{p}+\xi^*]$ $F^*(p) \geq G^*(p) \geq \frac{p}{2\bar{p}}$. Since f is convex, the right hand side of (6.1) is greater than $f'(\frac{\bar{p}+\xi}{2\bar{p}^*})/f'(\frac{\bar{p}-\xi}{2\bar{p}})$. Define

$$I(\alpha) = \frac{f'(\alpha(\bar{p}+\xi))}{f'(\alpha(\bar{p}-\xi))}$$

In order to prove (6.1), it is sufficient to show that $I'(\alpha) \geq 0$. It is easy to verify that if the elasticity of f' is nondecreasing, then $I' > 0$.

Let $\bar{p} \geq \frac{1}{2}$. In this case, on $[\bar{p}-\xi^*,\bar{p}]$ $F^*(p) \leq G^*(p) \leq \frac{p+1-2\bar{p}}{2-2\bar{p}}$, and on $[\bar{p},\bar{p}+\xi^*]$ these inequalities are reversed. As before, the right-hand side of (6.1) is greater than $f'(\frac{1-\bar{p}+\xi}{2-2\bar{p}})/f'(\frac{1-\bar{p}-\xi}{2-2\bar{p}})$. Let $\bar{q} = 1 - \bar{p}$. It is thus sufficient to prove that

$$\frac{\bar{f}'(\bar{q}+\xi)}{\bar{f}'(\bar{q}-\xi)} \geq \frac{\bar{f}'\ \frac{\bar{q}+\xi}{2\bar{q}}}{\bar{f}'\ \frac{\bar{q}-\xi}{2\bar{q}}}$$

which holds if the elasticity of $\bar{f}$ is nonincreasing.

If $x < 0$, the proof is similar.

Proof of Theorem 7.1:

Let $x > 0$. $\frac{y}{\bar{p}x} = \frac{1}{\bar{p}} \int_{\bar{p}-\xi^*}^{\bar{p}+\xi^*} f(z)f'(1-F^*(z-\bar{p}+\xi^*))F^{*\prime}(z-\bar{p}+\xi^*)dz =$
$\frac{1}{\bar{p}} \int_{-\xi^*}^{\xi^*} f(z+\bar{p})f'(1-F^*(z+\xi^*))F^{*\prime}(z+\xi^*)dz$. $\frac{\partial}{\partial \varepsilon} \frac{y}{\bar{p}x} = \frac{\partial}{\partial \bar{p}} \frac{y}{\bar{p}x}$, hence $\frac{\partial}{\partial \varepsilon} \frac{y}{\bar{p}x} > 0$ iff

$$\int_{-\xi^*}^{\xi^*} [\bar{p}f'(z+\bar{p}) - f(z+\bar{p})]f'(1-F^*(z+\xi^*))F^{*\prime}(z+\xi^*)dz > 0.$$

Since F^* is symmetric around ξ^*, it is sufficient to prove that for every $0 < z < \xi^*$,

$$[\bar{p}f'(\bar{p}+z) - f(\bar{p}+z)]f'(1-F^*(\xi^*+z)) + [\bar{p}f'(\bar{p}-z) - f(\bar{p}-z)]f'(1-F^*(\xi^*-z)) > 0.$$

f is convex, hence its elasticity is greater than 1. It is therefore sufficient to prove that

$$f'(\bar{p}-z)f'(1-F^*(\xi^*-z)) > f'(\bar{p}+z)f'(1-F^*(\xi^*+z)) \iff$$

$$\frac{f'(\bar{p}+z)}{f'(\bar{p}-z)} < \frac{f'(1-F^*(\xi^*-z))}{f'(1-F^*(\xi^*+z))}$$

As was proved in the proof of Theorem 6.1, this inequality holds true if $H_{\xi^*} >_a F^*$.

If $x < 0$, the proof is similar.

Proof of Theorem 7.2

If f is convex and if $f(0) = 0$, then $f'(p) > \frac{f(p)}{p}$, hence

$$\psi'(\delta) = \frac{\delta\alpha^* f'(\delta\alpha^*) - f(\delta\alpha^*)}{\delta^2 \bar{p}} + \frac{1}{\bar{p}} \int_{\alpha^*}^{\beta^*} zf''(\delta z)f(1-F^*(z))dz > 0$$

A similar proof holds for $\bar{\psi}$.

Proof of Theorem 8.1

Let $x > 0$. In (3.2) it is assumed that f is differentiable. If f is everywhere continuous and everywhere differentiable except for the points

$a_1,\ldots,a_k \in (\alpha^*,\beta^*)$, then the value of the ambiguous lottery $(x,S;0,\sim S)$ is

$$u(x)f(\alpha^*) + u(x) \lim_{\varepsilon\to 0} \sum_{i=1}^{k+1} \int_{a_{i-1}+\varepsilon}^{a_i-\varepsilon} f'(z)f(1-F^*(z))dz \tag{8.1}$$

where $a_0 = \alpha^*$ and $a_{k+1} = \beta^*$.

Assume without loss of generality that $u(x) = 1$ and let $\Phi(f)$ denote the value of $(x,S;0,\sim S)$ under f. Let $\alpha^* \leq \alpha < \alpha + \gamma \leq \beta^*$ and let

$$g_{\alpha,\gamma}(p) = \begin{cases} g(p) & p \leq \alpha \\ [\frac{g(\alpha+\gamma)-g(\alpha)}{\gamma}]p + \frac{\gamma g(\alpha)-\alpha g(\alpha+\gamma)+\alpha g(\alpha)}{\gamma} & \alpha \leq p \leq \alpha+\gamma \\ g(p) & p \geq \alpha+\gamma \end{cases}$$

Obviously, $g_{\alpha,\gamma}(p) \geq g(p)$ for all p. By (8.1),

$$\Phi(g_{\alpha,\gamma}) - \Phi(g) \geq \int_{\alpha}^{\alpha+\gamma} [\frac{g(\alpha+\gamma)-g(\alpha)}{\gamma} - g'(z)]g(1-F^*(z))dz = \phi(\gamma)$$

$\lim_{\gamma\to 0} \phi(\gamma) = 0$, and ϕ is increasing with γ. Indeed,

$$\phi'(\gamma) = \frac{1}{\gamma}[g'(\alpha+\gamma) - \frac{g(\alpha+\gamma)-g(\alpha)}{\gamma}][\int_{\alpha}^{\alpha+\gamma} g(1-F^*(z))dz - \gamma g(1-F^*(\alpha+\gamma))]$$

which is positive, since g is convex and g and F^* are increasing. It thus follows that there is a convex differentiable function $\tilde{g}$ such that for every p, $\tilde{g}(p) \geq g_{\alpha,\gamma}(p)$ and $\Phi(\tilde{g}) \geq \Phi(g)$. By repeating this process $[\frac{1}{\varepsilon}] + 1$ times, one can find for every $\varepsilon > 0$ a convex and differentiable function f_ε such that for every p, $g(p) \leq f_\varepsilon(p) \leq f(p)$, $f(p) - f_\varepsilon(p) \leq \varepsilon$, and $\Phi(f_\varepsilon) \geq \Phi(g)$. Obviously, $\Phi(f) - \Phi(f_\varepsilon) < \varepsilon$, hence $\Phi(f) \geq \Phi(g)$. Since both decision makers have the same utility function, I is willing to pay more than II for this lottery.

If $x < 0$, then $f > g$ implies $\bar{f} < \bar{g}$ and the proof is similar.

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