

## Exercise 6

- 1) Read the paper "Modeling and pricing long memory in stock market volatility" by Bollerslev and Mikkelsen.
- 2) For the model:

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$$

$$\varepsilon_t = u_t(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2)^{.5}$$

where  $u_t \sim IIN(0, 1)$

- i) Derive  $Var(\varepsilon_t)$  and  $Var(\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots)$

Which assumption(s) do you need to make to ensure that  $Var(\varepsilon_t)$  exists?

- ii) Derive  $Var(y_t)$

- 3) Consider the following model

$$y_t = \mu + \varepsilon_t$$

$$\varepsilon_t = u_t(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^{.5}$$

Calculate the third and fourth moments of  $y_t$ .

- 4) Show that

- i) for the following GARCH(1,1)

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \text{ we get:}$$

$$E(\varepsilon_{t+j}^2 | I_{t-1}) = \frac{w}{1-(\alpha+\beta)} + (\sigma_t^2 - \frac{w}{1-(\alpha+\beta)})(\alpha + \beta)^j$$

- ii) for an IGARCH(1,1)

$$E(\varepsilon_{t+j}^2 | I_{t-1}) = Jw + \sigma_t^2$$

- 5) Using Eviews (file named return.wf1),

- (a) find the Preferred GARCH(p,q) model for the series "returns".
- (b) Explain whether you find evidence of leverage effects.
- (c) Explain whether you think Garch-M models are appropriate for this data.

## Solution

2) Consider the Model

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$$

$$\varepsilon_t = u_t(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2)^{.5} \quad \text{where } u_t \sim IIN(0, 1)$$

i)  $Var(\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots)$

First notice that

$$E(\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) = (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2)^{.5} E(u_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) = 0$$

since  $u_t \sim IIN(0, 1)$ . Using iterative expectations we also know that  $E(\varepsilon_t) = 0$ .

- (Iterative expectations state that  $E(x_t) = E(E(x_t | x_{t-1}, x_{t-2}, x_{t-3}, \dots)))$ )

Then

$$\begin{aligned} Var(\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) &= E(\varepsilon_t^2 | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) \\ &= (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2) E(u_t^2 | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) \\ &= (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2) \end{aligned}$$

since  $E(u_t^2 | \varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, \dots) = 1$ .

Using *Iterative expectations*  $V(\varepsilon_t) = E(\varepsilon_t^2) = E(E(\varepsilon_t^2 | I_{t-1}))$  we can easily find the unconditional variance.

$$\begin{aligned} E(\varepsilon_t^2) &= E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \alpha_3 \varepsilon_{t-3}^2 + \alpha_4 \varepsilon_{t-4}^2) \\ &= \alpha_0 + \alpha_1 E(\varepsilon_{t-1}^2) + \alpha_2 E(\varepsilon_{t-2}^2) + \alpha_3 E(\varepsilon_{t-3}^2) + \alpha_4 E(\varepsilon_{t-4}^2). \end{aligned}$$

If  $\varepsilon_t^2$  is covariance stationary, i.e. if  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1$ , then  $E(\varepsilon_t^2) = \frac{\alpha_0}{1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}$ .

ii) To derive  $Var(y_t)$  we need to calculate the autocovariance functions and substitute  $\sigma_\varepsilon^2$  by  $\frac{\alpha_0}{1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}$ .

3) For the following model

$$\begin{aligned} y_t &= \mu + \varepsilon_t \\ \varepsilon_t &= u_t(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^{.5} \quad \text{where } u_t \sim IIN(0, 1) \end{aligned}$$

The third moment is

$$E(y_t - \mu)^3 = E(E(\varepsilon_t^3 | I_{t-1})) = 0 \text{ since } E(\varepsilon_t^3 | I_{t-1}) = (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^{1.5} E(u_t^3 | I_{t-1}) = 0 \text{ because the Normal distribution is symmetric.}$$

The fourth moment is

$$E(y_t - \mu)^4 = E(E(\varepsilon_t^4 | I_{t-1}))$$

Notice that

$$E(\varepsilon_t^4 | I_{t-1}) = (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^2 E(u_t^4 | I_{t-1}) = 3(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^2 \text{ because the Normal distribution's kurtosis is 3.}$$

To calculate the fourth moment we simply apply iterative expectations

$$E(\varepsilon_t^4) = 3(\alpha_0^2 + 2\alpha_0\alpha_1 E(\varepsilon_{t-1}^2) + \alpha_1^2 E(\varepsilon_{t-1}^4))$$

If  $3\alpha_1^2 < 1$ , then

$$E(y_t - \mu)^4 = E(\varepsilon_t^4) = \frac{3(\alpha_0^2 + 2\alpha_0\alpha_1 E(\varepsilon_{t-1}^2))}{1 - 3\alpha_1^2} = \frac{3(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1 - \alpha_1})}{1 - 3\alpha_1^2}$$

4) Consider the following GARCH Model

$$\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

i) Rewrite this expression using the fact the definition of an innovation  $\nu_t = \varepsilon_t^2 - E(\varepsilon_t^2 | I_{t-1}) = \varepsilon_t^2 - \sigma_t^2$ .

$$\varepsilon_t^2 = w + (\alpha + \beta) \varepsilon_{t-1}^2 - \beta \nu_{t-1} + \nu_t$$

Conditioning in Information at time t-2 we get

$$E(\varepsilon_t^2 | I_{t-2}) = w + (\alpha + \beta) E(\varepsilon_{t-1}^2 | I_{t-2})$$

Leading one period

$$E(\varepsilon_{t+1}^2 | I_{t-1}) = w + (\alpha + \beta) E(\varepsilon_t^2 | I_{t-1})$$

Conditioning in Information at time t-2 we get

$$E(\varepsilon_{t+1}^2 | I_{t-2}) = w + (\alpha + \beta) E(\varepsilon_t^2 | I_{t-2})$$

Substituting backwards

$$\begin{aligned}
E(\varepsilon_{t+1}^2|I_{t-2}) &= w + (\alpha + \beta)(w + (\alpha + \beta)E(\varepsilon_{t-1}^2|I_{t-2})) \\
&= (1 + (\alpha + \beta))w + (\alpha + \beta)^2 E(\varepsilon_{t-1}^2|I_{t-2})
\end{aligned}$$

Leading one period

$$E(\varepsilon_{t+2}^2|I_{t-1}) = (1 + (\alpha + \beta))w + (\alpha + \beta)^2 E(\varepsilon_t^2|I_{t-1})$$

Repeating the procedure we obtain

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + (\alpha + \beta) + (\alpha + \beta)^2 + \dots + (\alpha + \beta)^{j-1})w + (\alpha + \beta)^j E(\varepsilon_t^2|I_{t-1})$$

summing we get

$$E(\varepsilon_{t+j}^2|I_{t-1}) = \frac{1 - (\alpha + \beta)^j}{1 - (\alpha + \beta)} w + (\alpha + \beta)^j E(\varepsilon_t^2|I_{t-1})$$

rearranging terms

$$E(\varepsilon_{t+j}^2|I_{t-1}) = \frac{w}{1 - (\alpha + \beta)} + (E(\varepsilon_t^2|I_{t-1}) - \frac{w}{1 - (\alpha + \beta)})(\alpha + \beta)^j.$$

ii) To derive the formulae use

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + (\alpha + \beta) + (\alpha + \beta)^2 + \dots + (\alpha + \beta)^{j-1})w + (\alpha + \beta)^j E(\varepsilon_t^2|I_{t-1})$$

and substitute  $\alpha + \beta = 1$ .

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + 1 + 1 + \dots + 1)w + (1)^j E(\varepsilon_t^2|I_{t-1}) = Jw + E(\varepsilon_t^2|I_{t-1}).$$