Exercise 7

1) Read the paper, "Rational-Expectations Econometric Analysis of Changes in Regime. An investigation of the Term Structure of Interest Rates". James Hamilton.

2) Consider the following representation of a Markov process.

$$\begin{bmatrix} 1-x_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} 1-x_t \\ x_t \end{bmatrix} + \begin{bmatrix} \zeta_{1,t+1} \\ \zeta_{2,t+1} \end{bmatrix}$$

or:

$$W_{t+1} = PW_t + U_{t+1}$$

where

$W_t = \begin{bmatrix} 1-x_{t+1} \\ x_{t+1} \end{bmatrix}$ and $U_{t+1} = \begin{bmatrix} \zeta_{1,t+1} \\ \zeta_{2,t+1} \end{bmatrix}$

and $E_t\zeta_{1,t+1} = 0, E_t = \zeta_{2,t+1}$, so $E_tU_{t+1} = 0$

i) Which values $\zeta_{1,t+1}$ and $\zeta_{2,t+1}$ should take at t+1 (for $x_t = 1$ and for $x_t = 0$) to ensure that, $E_t\zeta_{1,t+1} = 0, E_t = \zeta_{2,t+1}$.

3)

i) Find the eigen-values, λ_1 and λ_2 of the transition probability Matrix defined as in two.

ii) Find the associated eigen-vectors.

iii) Show that P can be written as $P = T\Lambda T^{-1}$,where T is the matrix of eigen-vectors and Λ is a diagonal matrix of eigen-values.

iv) Show that $P^n = T\Lambda^n T^{-1}$ and find P^n.

Solution

For the following representation of a Markov processes

$$\begin{bmatrix} 1-x_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} 1-x_t \\ x_t \end{bmatrix} + \begin{bmatrix} \zeta_{1,t+1} \\ \zeta_{2,t+1} \end{bmatrix},$$

we can use the second row to derive the properties of the shocks,i.e.,

$$x_{t+1} = (1-q)(1-x_t) + px_t + \zeta_{2,t+1}$$

Then conditional on $x_t = 0$,

If $x_{t+1} = 0$, $\zeta_{2,t+1} = -(1-q)$ with probability q

If $x_{t+1} = 1$, $\zeta_{2,t+1} = q$ with probability $1-q$.

Conditional on $x_t = 1$,

If $x_{t+1} = 0$, $\zeta_{2,t+1} = -p$ with probability $1-p$

If $x_{t+1} = 1$, $\zeta_{2,t+1} = 1-p$ with probability p

Then it is clear that $E(\zeta_{2,t+1}|x_t = 0) = E(\zeta_{2,t+1}|x_t = 1) = 0$.

Notice that using the law of iterative expectations we obtain that $E(\zeta_{2,t+1}) = 0$.

3)

i) To find the eigen values we solve

$$Det \begin{bmatrix} q-\lambda & (1-p) \\ (1-q) & p-\lambda \end{bmatrix} = 0.$$

which is equivalent to solve $\lambda^2 - (p+q)\lambda + (-1+p+q) = 0$, which gives $\lambda_1 = 1$ and $\lambda_2 = (-1+p+q)$.

ii) The associated eigenvectors are calculated using the formulae:

$Px = \lambda x$, where P is the transition matrix and x is the eigen vector associated to λ.

Then for $\lambda_1 = 1$.

$$\begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = 1 \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$$

which gives $qx_1^1 + (1-p)x_2^1 = x_1^1$, or $x_1^1 = \frac{1-p}{1-q}x_2^1$. Assigning the value $\frac{1-q}{2-(p+q)}$ to x_2^1. gives $x_1^1 = \frac{1-p}{2-(p+q)}$.

Then for $\lambda_1 = 1$ the associated eigenvector is $\begin{bmatrix} \frac{1-p}{2-(p+q)}. \\ \frac{1-q}{2-(p+q)}. \end{bmatrix}$.

For $\lambda_2 = (-1 + p + q)$.

$$\begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = (-1 + p + q) \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix}$$

which gives $qx_1^2 + (1-p)x_2^2 = (-1+p+q)x_1^2$, or $x_1^1 = -x_2^2$. Assigning the value 1 to x_2^2. gives $x_1^1 = -1$.

Then for $\lambda_1 = (-1 + p + q)$ the associated eigenvector is $\begin{bmatrix} -1 \\ 1. \end{bmatrix}$.

iii) Notice that we can write both results using matrix notation as

$$\begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} \frac{1-p}{2-(p+q)} & -1 \\ \frac{1-q}{2-(p+q)} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1-p}{2-(p+q)} & -1 \\ \frac{1-q}{2-(p+q)} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 + p + q \end{bmatrix}.$$

If we define $T = \begin{bmatrix} \frac{1-p}{2-(p+q)} & -1 \\ \frac{1-q}{2-(p+q)} & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 + p + q \end{bmatrix}$, then $PT = T\Lambda$ or $P = T\Lambda T^{-1}$.

iv) $P^n = T\Lambda T^{-1} T\Lambda T^{-1} T\Lambda T^{-1} .... T\Lambda T^{-1} = T\Lambda^n T^{-1}$.

If we substitute these values we obtained the expression derived in the lecture notes.