Exercise 6

1) Read the paper "Modeling and pricing long memory in stock market volatility" by Bollerslev and Mikkelsen.

2) For the model:

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$$
$$\varepsilon_t = u_t(\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2)^{.5}$$
$$where\ u_t\tilde{}IIN(0,1)$$

i) Derive $Var(\varepsilon_t)$ and $Var(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...)$

Which assumption(s) do you need to make to ensure that $Var(\varepsilon_t)$ exists?

ii) Derive $Var(y_t)$

3) Consider the following model

$$y_t = \mu + \varepsilon_t$$
$$\varepsilon_t = u_t(\alpha_0 + \alpha_1\varepsilon_{t-1}^2)^{.5}$$

Calculate the third and fourth moments of y_t.

4) Show that

i) for the following GARCH(1,1)

$$\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \text{ we get:}$$
$$E(\varepsilon_{t+j}^2|I_{t-1}) = \frac{w}{1-(\alpha+\beta)} + (\sigma_t^2 - \frac{w}{1-(\alpha+\beta)})(\alpha+\beta)^j$$

ii) for an IGARCH(1,1)

$$E(\varepsilon_{t+j}^2|I_{t-1}) = Jw + \sigma_t^2$$

Solution

2) Consider the Model

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$$

$$\varepsilon_t = u_t(\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2)^{.5} \qquad where\ u_t\tilde{\ }IIN(0,1)$$

i) $Var(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...)$

First notice that

$$E(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) = (\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2)^{.5}E(u_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) = 0$$

since $u_t\tilde{\ }IIN(0,1)$. Using iterative expectations we also know that $E(\varepsilon_t) = 0$.

- *(Iterative expactations state that* $E(x_t) = E(E(x_t|x_{t-1}, x_{t-2}, x_{t-3}, ...))$)

Then

$$\begin{aligned} Var(\varepsilon_t|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) &= E(\varepsilon_t^2|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) \\ &= (\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2)E(u_t^2|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) \\ &= (\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2) \end{aligned}$$

since $E(u_t^2|\varepsilon_{t-1}, \varepsilon_{t-2}, \varepsilon_{t-3}, ...) = 1$.
Using $Iterative\ expactations$ $V(\varepsilon_t) = E(\varepsilon_t^2) = E(E(\varepsilon_t^2|I_{t-1}))$ we can easily find the unconditional variance.

$$\begin{aligned} E(\varepsilon_t^2) &= E(\alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \alpha_3\varepsilon_{t-3}^2 + \alpha_4\varepsilon_{t-4}^2) \\ &= \alpha_0 + \alpha_1 E(\varepsilon_{t-1}^2) + \alpha_2 E(\varepsilon_{t-2}^2) + \alpha_3 E(\varepsilon_{t-3}^2) + \alpha_4 E(\varepsilon_{t-4}^2). \end{aligned}$$

If ε_t^2 is covariance stationary, i.e. if $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1$, then $E(\varepsilon_t^2) = \dfrac{\alpha_0}{1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}$.

ii) To derive $Var(y_t)$ we need to calculate the autocovariance functions and substitute σ_ε^2 by $\dfrac{\alpha_0}{1-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}$.

3) For the following model

$$\begin{aligned} y_t &= \mu + \varepsilon_t \\ \varepsilon_t &= u_t(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)^{.5} \qquad where\ u_t\tilde{}IIN(0,1) \end{aligned}$$

The third moment is

$E(y_t-\mu)^3 = E(E(\varepsilon_t^3|I_{t-1})) = 0$ since $E(\varepsilon_t^3|I_{t-1}) = (\alpha_0+\alpha_1\varepsilon_{t-1}^2)^{1.5}E(u_t^3|I_{t-1}) = 0$ because the Normal distribution is symmetric.

The fourth moment is

$E(y_t-\mu)^4 = E(E(\varepsilon_t^4|I_{t-1}))$

Notice that

$E(\varepsilon_t^4|I_{t-1}) = (\alpha_0+\alpha_1\varepsilon_{t-1}^2)^2E(u_t^4|I_{t-1}) = 3(\alpha_0+\alpha_1\varepsilon_{t-1}^2)^2$because the Normal distribution´s kurtosis is 3.

To calculate the fourth moment we simply apply iterative expectations

$$E(\varepsilon_t^4) = 3(\alpha_0^2 + 2\alpha_0\alpha_1 E(\varepsilon_{t-1}^2) + \alpha_1^2 E(\varepsilon_{t-1}^4))$$

If $3\alpha_1^2 < 1, then$

$$E(y_t-\mu)^4 = E(\varepsilon_t^4) = \frac{3(\alpha_0^2 + 2\alpha_0\alpha_1 E(\varepsilon_{t-1}^2))}{1-3\alpha_1^2} = \frac{3(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1})}{1-3\alpha_1^2}$$

.

4) Consider the following GARCH Model

$$\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

i) Rewrite this expression using the fact the definition of an innovation $\nu_t = \varepsilon_t^2 - E(\varepsilon_t^2|I_{t-1}) = \varepsilon_t^2 - \sigma_t^2$.

$$\varepsilon_t^2 = w + (\alpha+\beta)\varepsilon_{t-1}^2 - \beta\nu_{t-1} + \nu_t$$

Conditioning in Information at time t-2 we get

$$E(\varepsilon_t^2|I_{t-2}) = w + (\alpha+\beta)E(\varepsilon_{t-1}^2|I_{t-2})$$

Leading one period

$$E(\varepsilon_{t+1}^2|I_{t-1}) = w + (\alpha+\beta)E(\varepsilon_t^2|I_{t-1})$$

Conditioning in Information at time t-2 we get

$$E(\varepsilon_{t+1}^2|I_{t-2}) = w + (\alpha+\beta)E(\varepsilon_t^2|I_{t-2})$$

Substituting backwards

$$\begin{aligned} E(\varepsilon_{t+1}^2|I_{t-2}) &= w + (\alpha+\beta)(w + (\alpha+\beta)E(\varepsilon_{t-1}^2|I_{t-2})) \\ &= (1 + (\alpha+\beta))w + (\alpha+\beta)^2 E(\varepsilon_{t-1}^2|I_{t-2}) \end{aligned}$$

Leading one period

$$E(\varepsilon_{t+2}^2|I_{t-1}) = (1 + (\alpha+\beta))w + (\alpha+\beta)^2 E(\varepsilon_t^2|I_{t-1})$$

Repeating the procedure we obtain

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + (\alpha+\beta) + (\alpha+\beta)^2 + \ldots.. + (\alpha+\beta)^{j-1})w + (\alpha+\beta)^j E(\varepsilon_t^2|I_{t-1})$$

summing we get

$$E(\varepsilon_{t+j}^2|I_{t-1}) = \frac{1-(\alpha+\beta)^j}{1-(\alpha+\beta)}w + (\alpha+\beta)^j E(\varepsilon_t^2|I_{t-1})$$

rearranging terms

$$E(\varepsilon_{t+j}^2|I_{t-1}) = \frac{w}{1-(\alpha+\beta)} + (E(\varepsilon_t^2|I_{t-1}) - \frac{w}{1-(\alpha+\beta)})(\alpha+\beta)^j.$$

ii) To derive the formulae use

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + (\alpha+\beta) + (\alpha+\beta)^2 + ... + (\alpha+\beta)^{j-1})w + (\alpha+\beta)^j E(\varepsilon_t^2|I_{t-1})$$

and substitute $\alpha + \beta = 1$.

$$E(\varepsilon_{t+j}^2|I_{t-1}) = (1 + 1 + 1 + ... + 1)w + (1)^j E(\varepsilon_t^2|I_{t-1}) = Jw + E(\varepsilon_t^2|I_{t-1}).$$