Exercise 4

1)

a) Derive the variance ratio:

$$\lambda_1(k) = \frac{Var(\Delta_k y_t)}{Var(\Delta y_t)}$$

for:

i) $y_t = \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

ii) $y_t = y_{t-1} + \varepsilon_t$

b) How can you tell from a plot of the variance ratio if the process has a unit root?

2) Consider the following example:

$$y_t + \beta x_t = u_{1t}$$
$$y_t + \alpha x_t = u_{2t}$$

where:

$$u_{1t} = 0.2u_{1t-1} + 0.8u_{1t-1} + \varepsilon_{1t}$$
$$u_{2t} = \rho u_{2t-1} + 0.5u_{2t-1} + \varepsilon_{2t}$$

i) What is the order of integration of y_t and x_t?

ii) Under which conditions are y_t and x_t cointegrated?

iii) Find the MA and ECM representation (assuming that x and y are cointegrated)

Solution

1) Consider the following ARMA(1,1)

$$y_t = \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

To derive the variance ratio $\lambda_1(k) = \dfrac{Var(\Delta_k y_t)}{Var(\Delta y_t)}$ we need to substitute backward until a pattern which will enable us to compute the k´th difference is apparent.

Substituting backwards we get

$$y_t = \phi_1(\phi_1 y_{t-2} + \theta_1 \varepsilon_{t-2} + \varepsilon_{t-1}) + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

or rearranging terms

$$y_t = \phi_1^2 y_{t-2} + \phi_1 \theta_1 \varepsilon_{t-2} + (\phi_1 + \theta_1)\varepsilon_{t-1} + \varepsilon_t$$

Substituting again we get

$$y_t = \phi_1^3 y_{t-3} + \phi_1^2 \theta_1 \varepsilon_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \theta_1 \varepsilon_{t-2} + (\phi_1 + \theta_1)\varepsilon_{t-1} + \varepsilon_t$$

or rearranging terms

$$y_t = \phi_1^3 y_{t-3} + \phi_1^2 \theta_1 \varepsilon_{t-3} + \phi_1(\phi_1 + \theta_1)\varepsilon_{t-2} + (\phi_1 + \theta_1)\varepsilon_{t-1} + \varepsilon_t$$

Substituting again we get

$$y_t = \phi_1^4 y_{t-4} + \phi_1^3 \theta_1 \varepsilon_{t-4} + \phi_1^3 \varepsilon_{t-3} + \phi_1^2 \theta_1 \varepsilon_{t-3} + \phi_1(\phi_1 + \theta_1)\varepsilon_{t-2} + (\phi_1 + \theta_1)\varepsilon_{t-1} + \varepsilon_t$$

or rearranging terms

$$y_t = \phi_1^4 y_{t-4} + \phi_1^3 \theta_1 \varepsilon_{t-4} + \phi_1^2(\phi_1 + \theta_1)\varepsilon_{t-3} + \phi_1(\phi_1 + \theta_1)\varepsilon_{t-2} + (\phi_1 + \theta_1)\varepsilon_{t-1} + \varepsilon_t$$

At this stage should be clear that is we substitute backwards k-1 times we should end with the following expression:

$$y_t = \phi_1^k y_{t-k} + \phi_1^{k-1}\theta_1\varepsilon_{t-k} + \phi_1^{k-2}(\phi_1+\theta_1)\varepsilon_{t-(k-1)} + ... + \phi_1(\phi_1+\theta_1)\varepsilon_{t-2} + (\phi_1+\theta_1)\varepsilon_{t-1} + \varepsilon_t$$

Now it is straight forward to calculate the variance of the k´the difference as

$$\begin{aligned} Var(\Delta_k y_t) = Var((\phi_1^k - 1)y_{t-k} + \phi_1^{k-1}\theta_1\varepsilon_{t-k} + \phi_1^{k-2}(\phi_1+\theta_1)\varepsilon_{t-(k-1)} + ... \\ ..... + \phi_1(\phi_1+\theta_1)\varepsilon_{t-2} + (\phi_1+\theta_1)\varepsilon_{t-1} + \varepsilon_t) \end{aligned}$$

Notice that even though y_{t-k} is uncorrelated with $(\varepsilon_{t-(k-1)}, ..., \varepsilon_{t-2}, \varepsilon_{t-1}, \varepsilon_t)$, it is clearly correlated with ε_{t-k}.

Then we can consider the variance as the sum of three terms

a)

$$Var((\phi_1^k - 1)y_{t-k} + \phi_1^{k-1}\theta_1\varepsilon_{t-k}) = (\phi_1^k - 1)^2 Var(y_{t-k}) + \phi_1^{2(k-1)}\theta_1^2\sigma_\varepsilon^2 + 2(\phi_1^k - 1)\phi_1^{k-1}\theta_1\sigma_\varepsilon^2$$

b)

$$\begin{aligned} Var(\phi_1^{k-2}(\phi_1+\theta_1)\varepsilon_{t-(k-1)} + ... + \phi_1(\phi_1+\theta_1)\varepsilon_{t-2} + (\phi_1+\theta_1)\varepsilon_{t-1}) &= (\phi_1+\theta_1)^2(\sum_{i=0}^{k\_2}\phi_1^{2i})\sigma_\varepsilon^2 \\ &= (\phi_1+\theta_1)^2\frac{1-\phi_1^{2(k-1)}}{1-\phi_1^2}\sigma_\varepsilon^2 \end{aligned}$$

c)

$$Var(\varepsilon_t) = \sigma_\varepsilon^2$$

Then the variance of $Var(\Delta_k y_t)$ can be written as

$$Var(\Delta_k y_t) = (\phi_1^k - 1)^2 Var(y_{t-k}) + \phi_1^{2(k-1)}\theta_1^2\sigma_\varepsilon^2 + 2(\phi_1^k - 1)\phi_1^{k-1}\theta_1\sigma_\varepsilon^2 + (\phi_1+\theta_1)^2\frac{1-\phi_1^{2(k-1)}}{1-\phi_1^2}\sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

Notice that the results depend on $Var(y_{t-k})$ which need to be calculated. Using the autocovariance function this is straight forward.

For the above ARMA(1,1)

$$\gamma(k) = E(y_t y_{t-k}) = \phi_1 E(y_{t-1}y_{t-k}) + \theta_1 E(\varepsilon_{t-1}y_{t-k}) + E(\varepsilon_t y_{t-k})$$

$$\gamma(0) = \phi_1\gamma(1) + \theta_1 E(\varepsilon_{t-1}y_t) + E(\varepsilon_t y_t) \qquad for\ k = 0.$$

which can be written as

$$\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \theta_1 E(\varepsilon_{t-1}(\phi_1 y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t)) \\ + E(\varepsilon_t(\phi_1 y_{t-1} + \theta_1\varepsilon_{t-1} + \varepsilon_t)) \qquad for\ k = 0.$$

which simplifies to

$$\gamma(0) = \phi_1\gamma(1) + \theta_1(\phi_1 + \theta_1)\sigma_\varepsilon^2 + \sigma_\varepsilon^2 \qquad for\ k = 0.$$

analogously we can find that

$$\gamma(1) = \phi_1\gamma(0) + \theta_1\sigma_\varepsilon^2 \qquad for\ k = 1.$$

Then

$$Var(y_{t-k}) = Var(y_t) = \gamma(0) = \frac{1 + \theta_1^2 + 2\phi_1\theta_1}{1 - \phi_1^2}\sigma_\varepsilon^2.$$

The the final expression for $Var(\Delta_k y_t)$ is

$$Var(\Delta_k y_t) = (\phi_1^k - 1)^2 \frac{1 + \theta_1^2 + 2\phi_1\theta_1}{1 - \phi_1^2}\sigma_\varepsilon^2 + \phi_1^{2(k-1)}\theta_1^2\sigma_\varepsilon^2 + \\ 2(\phi_1^k - 1)\phi_1^{k-1}\theta_1\sigma_\varepsilon^2 + (\phi_1 + \theta_1)^2 \frac{1 - \phi_1^{2(k-1)}}{1 - \phi_1^2}\sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

The expression for $Var(\Delta y_t)$ can be obtained simply by substituting k=1 in the expression above.

$$Var(\Delta y_t) = (\phi_1 - 1)^2 \frac{1 + \theta_1^2 + 2\phi_1\theta_1}{1 - \phi_1^2}\sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 + 2(\phi_1 - 1)\theta_1\sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

Then $\lambda_1(k) = \dfrac{Var(\Delta_k y_t)}{Var(\Delta y_t)}$ can be obtained by substituting the above formulae in this expression.

a)ii) For the variance ratio of a random walk see the lecture notes.

b) We have seen in the lecture that for a random walk

$$lim_{k\to\infty}\lambda_1(k) = lim_{k\to\infty}\frac{Var(\Delta_k y_t)}{Var(\Delta y_t)} = \infty$$

The result for an ARMA(1,1) is

$$lim_{k\to\infty}\lambda_1(k) = \frac{\frac{1+\theta_1^2+2\phi_1\theta_1}{1-\phi_1^2}\sigma_\varepsilon^2 + (\phi_1+\theta_1)^2\frac{1}{1-\phi_1^2}\sigma_\varepsilon^2 + \sigma_\varepsilon^2}{(\phi_1-1)^2\frac{1+\theta_1^2+2\phi_1\theta_1}{1-\phi_1^2}\sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 + 2(\phi_1-1)\theta_1\sigma_\varepsilon^2 + \sigma_\varepsilon^2}$$

, a constant.

2) For the model

$$y_t + \beta x_t = u_{1t} \qquad u_{1t} = 0.2u_{1t-1} + 0.8u_{1t-2} + \varepsilon_{1t}$$

$$y_t + \alpha x_t = u_{2t} \qquad u_{2t} = \rho u_{2t-1} + 0.5u_{2t-2} + \varepsilon_{2t}$$

i) The order the integration can be found by noting that u_{1t} is $I(1)$ and u_{2t} is assumed to be $I(0)$.

Then it is easy to show that both y_t and x_t can be written as a linear combination of $I(0)$ and $I(1)$ processes which is clearly $I(1)$.

$$\begin{bmatrix} 1 & \beta \\ 1 & \alpha \end{bmatrix}\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} u_{1t} \\ u_{1t} \end{bmatrix}$$

or

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \frac{1}{\alpha-\beta}\begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_{1t} \\ u_{1t} \end{bmatrix}$$

Then both y_t and x_t are linear combinations of $I(0)$ and $I(1)$ processes which are $I(1)$.

ii) The processes y_t and x_t are cointegrated if all the roots of the polynomial $(1-\rho L - 0.5L^2)$ of the AR(2) for u_{2t} are outside the unit circle.

Necessary and sufficient conditions for stationarity are

$$\begin{aligned} \rho + 0.5 &< 1 \\ -\rho + 0.5 &< 1 \\ 0.5 &> -1 \end{aligned}$$

This is satisfied for $-0.5 < \rho < 0.5$.

iii) The MA representation can be found as follows

For

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Delta u_{1t} \\ \Delta u_{1t} \end{bmatrix}$$

notice that the polynomial $(1-0.2L-0.8L^2) = 0$ has a unit root, therefore, it can be written as $(1-L)(1+0.8L) = 0$. The process can be written as $(1-0.2L-0.8L^2)u_{1t} = \varepsilon_{1t}$ or $(1-L)(1+0.8L)u_{1t} = \varepsilon_{1t}$ which implies that $\Delta u_{1t} = (1+0.8L)^{-1}\varepsilon_{1t}$.

Analogously we can write $\Delta u_{2t} = (1-L)(1-\rho L - 0.5L^2)^{-1}\varepsilon_{2t.}$

Then substituting in the expression above we get

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (1+0.8L)^{-1}\varepsilon_{1t} \\ (1-L)(1-\rho L - 0.5L^2)^{-1}\varepsilon_{2t.} \end{bmatrix}$$

The ECM representation can be found as follows

Rewrite $u_{1t} = 0.2u_{1t-1} + 0.8u_{1t-2} + \varepsilon_{1t}$ as $\Delta u_{1t} = -0.8\Delta u_{1t-1} + \varepsilon_{1t}$ and $u_{2t} = \rho u_{2t-1} + 0.5u_{2t-2} + \varepsilon_{2t}$ as $\Delta u_{2t} = (\rho - 0.5)u_{2t-1} - 0.5\Delta u_{2t-1} + \varepsilon_{2t}$.

Then substituting we get

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -0.8\Delta u_{1t-1} + \varepsilon_{1t} \\ (\rho - 0.5)u_{2t-1} - 0.5\Delta u_{2t-1} + \varepsilon_{2t} \end{bmatrix}$$

or

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ (\rho - 0.5)u_{2t-1} \end{bmatrix} - \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.8\Delta u_{1t-1} \\ 0.5\Delta u_{2t-1} \end{bmatrix} + \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

Noting that $\Delta u_{1t-1} = \Delta y_{t-1} + \beta \Delta x_{t-1}$ and $\Delta u_{2t-1} = \Delta y_{t-1} + \alpha \Delta x_{t-1}$ we can write the ECM as

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ (\rho - 0.5) u_{2t-1} \end{bmatrix}$$
$$- \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.8(\Delta y_{t-1} + \beta \Delta x_{t-1}) \\ 0.5(\Delta y_{t-1} + \alpha \Delta x_{t-1}) \end{bmatrix} + \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

or

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ (\rho - 0.5) u_{2t-1} \end{bmatrix}$$
$$- \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0.8\beta \\ 0.5 & 0.5\alpha \end{bmatrix} \begin{bmatrix} \Delta y_{t-1} \\ \Delta x_{t-1} \end{bmatrix} + \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha & -\beta \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$