Economics 101A Notes on The Solution to Constrained Maximization Problems

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Abstract. This note discusses the Lagrange technique for the solution to constrained maximization problems and its extension to inequality conditions.

1. The technique of Lagrangian Multipliers

Suppose that x is an n dimensional vector and that $f(x)$ is a real valued function on x. Consider the following *constrained* maximization problem:

(Problem 1): maximize $f(x)$ subject to the constraint that $c - g(x) = 0$. Denote the solution by x^*.

The technique of Lagrange forms a new function L, called the Lagrangian, by adding a scalar λ multiplied by the constraint to the original function, i.e.

(Problem 2): $L(x, \lambda) = f(x) + \lambda(c - g(x))$

and finds the *unconstrained* maximum over x and minimum over λ for $L(x, \lambda)$. Suppose this solution say, $\widetilde{x}, \widetilde{\lambda}$ exists.

Now the first order conditions for a maximum of the function L tell us that the partial derivatives with respect to x_i for $i = 1, ..n$ and with respect to λ must all be zero, i.e.

$\frac{\partial L(x,\lambda)}{\partial x_i} = f_i(\widetilde{x}) - \lambda g_i(\widetilde{x}) = 0 \ \forall i = 1, ..., n$ and

$\frac{\partial L(x,\lambda)}{\partial \lambda} = c - g(\widetilde{x}) = 0.$

Since x^* solves Problem 1 and $\widetilde{x}$ satisfies the constraint, it immediately follows that $f(x^*) \geq f(\widetilde{x})$.

Since $\widetilde{x}, \widetilde{\lambda}$ maximizes the Lagrangian over x, then $L(\widetilde{x}, \widetilde{\lambda}) \geq L(x^*, \widetilde{\lambda})$, or, $f(\widetilde{x}) + \widetilde{\lambda}(c - g(\widetilde{x})) \geq f(x^*) + \widetilde{\lambda}(c - g(x^*))$

and since $c - g(x^*) = c - g(\widetilde{x}) = 0$, this implies $f(\widetilde{x}) \geq f(x^*)$.

Consequently, $f(\widetilde{x}) \equiv f(x^*) \equiv L(\widetilde{x}, \widetilde{\lambda})$.

Now the values $\widetilde{x}, \widetilde{\lambda}$ depend on the parameter c. That is, a change in c will alter $\widetilde{x}$ and $\widetilde{\lambda}$. To see the effect of this on the maximum value of $f(x^*)$, we may totally differentiate $L(\widetilde{x}, \widetilde{\lambda})$ with respect to c.

$\frac{dL(\widetilde{x},\widetilde{\lambda})}{dc} = \sum_i \frac{\partial L(\widetilde{x},\widetilde{\lambda})}{\partial \widetilde{x}_i} \frac{d\widetilde{x}_i}{dc} + \frac{\partial L(\widetilde{x},\widetilde{\lambda})}{\partial \widetilde{\lambda}} \frac{d\widetilde{\lambda}}{dc} + \frac{\partial L(\widetilde{x},\widetilde{\lambda})}{\partial c} = \frac{\partial L(\widetilde{x},\widetilde{\lambda})}{\partial c} = \widetilde{\lambda}.$

Thus $\widetilde{\lambda}$ measures the rate at which the *maximum* value of f is increased as the constraint is relaxed, i.e. c is increased.

2. Non-negativity constraint on the variables

Non-negativity constraints on the variables, i.e. $x_i \geq 0, i = 1, .., n$ can be easily dealt with in constrained and unconstrained problems.

The first order conditions, i.e. $\frac{\partial L(x,\lambda)}{\partial x_i} = f_i(\widetilde{x}) - \lambda g_i(\widetilde{x}) = 0 \forall i = 1, ..., n$, are simply replaced with inequalities

$\frac{\partial L(x,\lambda)}{\partial x_i} = f_i(\widetilde{x}) - \lambda g_i(\widetilde{x}) \leq 0 \forall i = 1, ..., n$ where $f_i(\widetilde{x}) - \lambda g_i(\widetilde{x}) < 0$ only if $x_i = 0$.

The interpretation is simple. If the derivative with respect to x_i were positive then we could increase the value of the Lagrangian by *raising* x_i. We could always do this without violating the non-negativity constraint and so we couldn't be at a maximum. Hence the weak inequality. If the derivative with respect to x_i were negative, then we could raise the value of the Lagrangian by *decreasing* x_i. If we are at a maximum then this cannot be so, i.e. it must be impossible to decrease x_i which could only mean that x_i was already zero.

3. Inequality Constraints

Finally, we can deal with constraints which respresent *weak inequalities*, i.e. $c - g(\widetilde{x}) \geq 0$ by using a *dummy* variable, say s, to turn it into an *equality* constraint as follows:

Let the constraint be written as $c - s - g(x) = 0$ and require that $s \geq 0$. Then, by the first order conditions, the derivative of the Lagrangian with respect to s must be non-positive and if it is strictly negative then $s = 0$. But the derivative $\frac{\partial L(x,s,\lambda)}{\partial s} = -\lambda$. So if $\lambda > 0$ (and consequently $\frac{\partial L(x,s,\lambda)}{\partial s} < 0$) then $s = 0$ and the constraint is satisfied with equality. If, on the other hand, the constraint is not binding (satisfied with slack), i.e. $s > 0$, then it must be the case that $\frac{\partial L(x,s,\lambda)}{\partial s} = -\lambda = 0$.

4. Interpretation of the Lagrangian Multiplier

When the maximiation is subject to the inequality constraint, $c - g(\widetilde{x}) \geq 0$, then the relationship above may be written as $\lambda(c - g(\widetilde{x})) = 0$.

Whenever the extreme solution is achieved in the interior of the constraint (the constraint is not *binding*) , i.e. $c - g(\widetilde{x}) > 0$, then the Lagrangian multiplier must be zero indicating that a weakening of the constraint would not contribute to the maximium value for objective.

When the solution is achieved on the boundary of the constraint, $c - g(\widetilde{x}) = 0$, then λ measures the value (in terms of the objective) of moving the constraint boundary by one unit.

In the case of utility maximization, where $f(x)$ is replaced by the utility function $U(x)$ and $c - g(x)$ is replaced by $y - \sum_i p_i x_i$, λ measures the marginal utility of income.