Economics 101A Notes on the Axiomatic Treatment of Risk

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Abstract

This note discusses the axioms for Expected Utility, the Allais and Ellsberg Paradoxes and derives the Expected Utility Theorem

0.1. Axioms on Lotteries and the Allais Paradox

A Lottery with mutually exclusive prizes denoted by a vector x received with probabilities denoted by a corresponding vector p will be written $L = (x, p)$

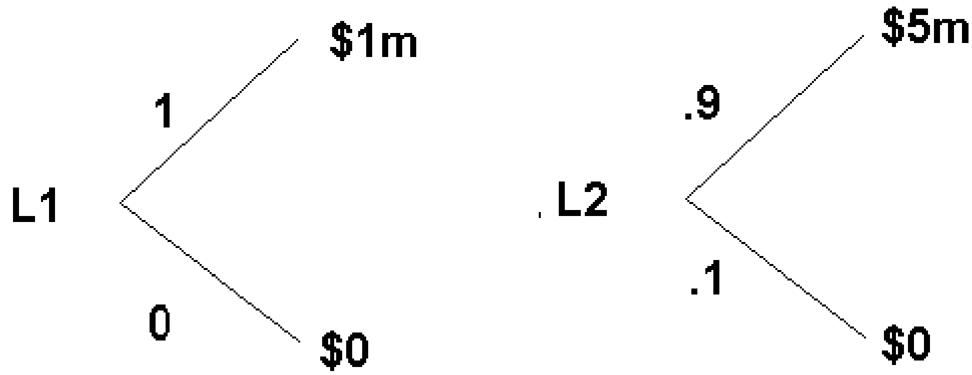
The following axioms are proposed:

1. Reduction of Compound Lotteries : A consumer's preferences over lotteries depends only upon the net probabilities of receiving the various prizes,
2. Transitivity: For any L, L', L'' if $L \succcurlyeq L'$ and $L' \succcurlyeq L''$ then $L \succcurlyeq L''$.
3. Continuity: For any L, L', L'' such that $L \succcurlyeq L' \succcurlyeq L''$, there exists $\alpha \epsilon [0, 1]$ and $\alpha L + (1 - \alpha)L'' \sim L'$.
4. Independence: If two lotteries differ only in one of their prizes, then the lotteries must be ordered in the same way as those prizes, i.e. $L \succcurlyeq L' \leftrightarrow \alpha L + (1 - \alpha)L'' \succcurlyeq \alpha L' + (1 - \alpha)L''$ where $\alpha \epsilon (0, 1]$.

The first three assumptions with the addition of some technical details are sufficient to prove the existence of a continuous utility function over the space of Lotteries.

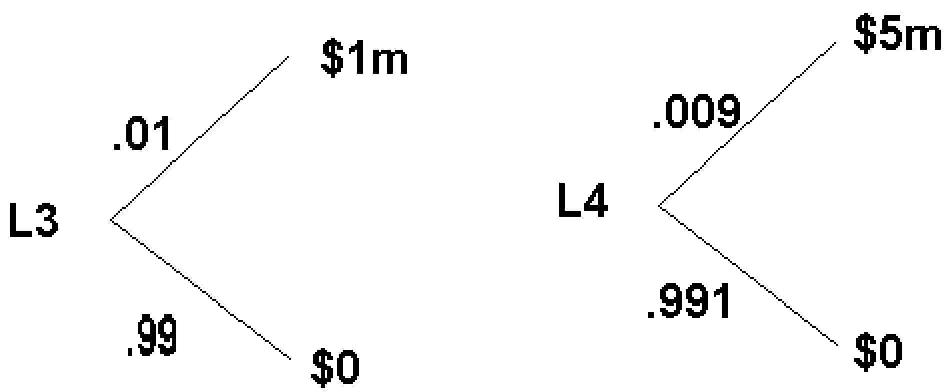
The fourth assumption implies that an expected utility function can be found to rationalize the preferences. This latter is not innocuous as the following version of the Allais paradox illustrates:

1. Consider a pair of Lotteries described by:

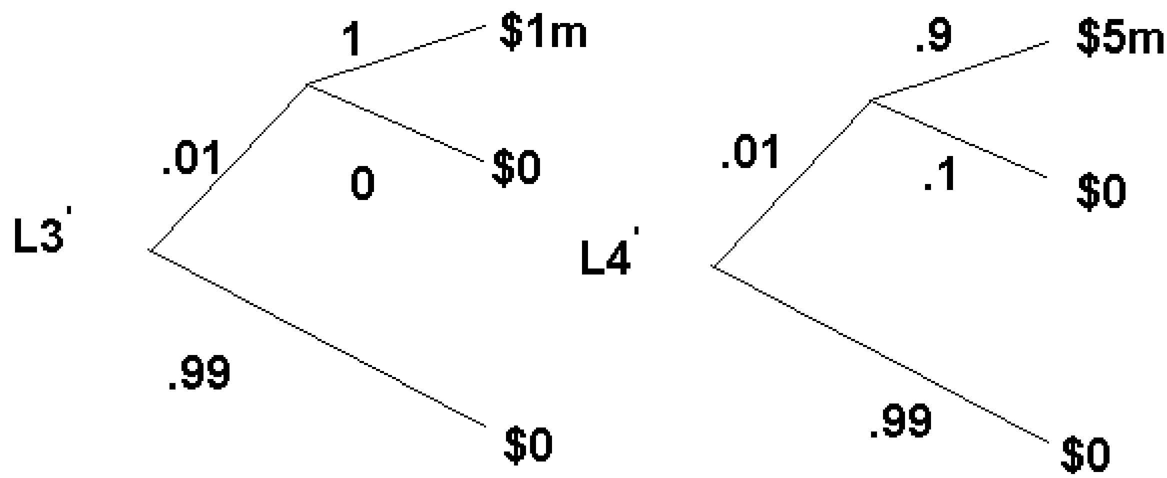


Most people might be expected to prefer $L1$ to $L2$ since the prospect of $1 million with certainty is more appealing than the risk of losing all possible gain with a 1/10th chance for the opportunity to gain $5 million even with a 9/10 chance.

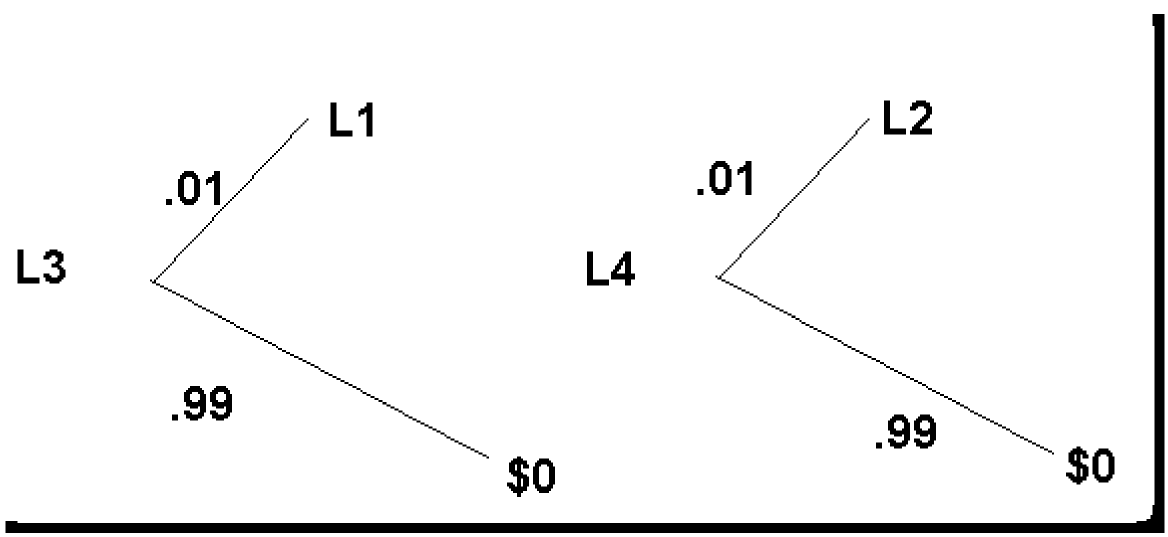
Consider now another pair of lotteries



Here, most would find $L4$ the more attractive since the probabilities appear almost the same as in $L3$ but with higher a higher payoff for winning. But $L3$ and $L4$ have the same probabilities of payoffs as $L3'$ and $L4'$.



which are simply



By the independence axiom, these must be ordered the same as $L1$ and $L2$.

0.2. Uncertainty

The term uncertainty (as compared to risk) is sometimes used to describe a situation in which the individual is unaware of the probabilities of events. Ellsberg presents the following example:

An individual is to draw a ball from either of two urns:

- the first urn contains 50 red balls and 50 blue balls.
- the second contains 100 balls some of which are red and the rest blue.
- the individual is asked to choose which of the urns to drawn from and then guess the color.
- a correct guess yields \$100, an incorrect one \$0.

A significant number of people prefer to guess from the urn whose probability distribution is known. Yet this is inconsistent with the assumption that such an individual has a subjective probability distribution over the color of the balls in the second urn. That is, since the individual is indifferent between red and black from the first urn where the probabilities are each 0.5 and a choice of red from the first urn is preferred to a choice of red from the second, the probability of red in the second must be less than 0.5. A similiar situation w.r.t. blue would mean that the sum of the probabilities of red and blue from the second urn must be less than 1!

0.3. The Expected Utility Theorem

Let b denote the best prize and w the worst. Arbitrarily, let $u(b) = 1$ and $u(w) = 0$. By continuity, for any prize z, there exists α s.t. $z \sim (w, b, (1 - \alpha), \alpha)$, the latter denoting the lottery with w as a prize with probability $(1 - \alpha)$ and b with probability α. Assign $u(z) = \alpha$. We may proceed in this manner to find the "utilities" of all possible prizes.

Consider now a lottery with, say, n prizes denoted by a vector x with corresponding probabilities of p another n vector whose elements are non-negative and sum to one. The *ith* prize, x_i, has utility $u(x_i)$ and is received with probability p_i.

Now x_i with certainty is equivalent to the lottery $(w, b, 1 - u(x_i), u(x_i))$. Replacing all of the prizes by their "w, b" equivalents produces an equivalent lottery

$$((w, b, 1 - u(x_1), u(x_1))\,(w, b, 1 - u(x_2), u(x_2))\,...\,(w, b, 1 - u(x_n), u(x_n))\,, p_1, p_2, ..., p_n)$$

(by the independence axiom).

Reducing that new lottery to simply the ultimate probabilities of w and b produces another equivalent lottery

$$\left(w, b, \sum p_i \left(1 - u(x_i)\right), \sum p_i u(x_i)\right)$$

(under the assumption on reduction of compound lotteries). But since $\sum p_i = 1$, this is simply

$$\left(w, b, \sum \left(1 - p_i u(x_i)\right), \sum p_i u(x_i)\right)$$

which is equivalent to a certain prize with utility $\sum p_i u(x_i)$.