Economics 101A Lecture Notes on Walrasian Equilibria

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Abstract. This note describes the proof of the existence of a Walrasian Equilibrium

The proof of the Existence of a Competitive Equilibrium is addressed in two parts:

0.1. Market Equilibrium. First, sufficient conditions for an equilibrium will be presented for the excess demand function $z(p)$. This vector valued function describes the excess of demand over supply, commodity by commodity, depending upon the prices prevailing in the economy. It is not necessary at this point to state more primitive assumptions that lead to these excess demand functions so long as the latter satisfy certain conditions:

Let $z(p)$ denote the market excess demand function and suppose that $z(p)$ satisfies the following:

1. $z(p)$ is continuous in p.
2. $pz(p) = 0$, Walras' law.
3. $z(p)$ is homogeneous of degree zero in p, i.e. $z(\lambda p) = z(p)$ for any positive scalar λ.

Then there exists p^* s.t. , $z(p^*) \leq 0$ and if $z_i(p^*) < 0$ then $p_i^* = 0$.

Define the following function

$$f_i(p) = \frac{p_i + \max\{0, z_i(p)\}}{1 + \sum_j \max\{0, z_j(p)\}}$$

Brouwer Fixed Point Theorem: If $f : S^{k-1} \to S^{k-1}$ is a continuous function from the unit simplex to itself, there is some $x \in S^{k-1}$ such that $x = f(x)$. The notation S^{k-1} denotes those non-negative k dimensional vectors whose components sum to 1.

Observations: Note first that if $p = (p_1, ..., p_N) \geq 0$ then $f = f(p) \geq 0$ as well.

Note second that $\sum_i f_i(p) = \frac{\sum_i p_i + \sum_i \max\{0, z_i(p)\}}{1 + \sum_j \max\{0, z_j(p)\}} = 1$ if $\sum p_i = 1$. So the function $f(p)$ maps a non-negative vector whose components sum to 1 into another non-negative vector whose components also sum to 1. Further, since the denominator, $1 + \sum_j \max\{0, z_j(p)\}$ is strictly positive and $\max\{0, z_i(p)\}$ is continuous is p, then $f(p)$ is also continuous in p.

By the *Brouwer Fixed Point Theorem*, there exists p^* such that $p^* = f(p^*)$.
Claim: At p^* , $z(p^*) \leq 0$ and if $z_i(p^*) < 0$ then $p_i = 0$.
Proof: By cross multiplication

$$p_i^* + p_i^* \sum_j \max\{0, z_j(p^*)\} = p_i^* + \max\{0, z_i(p^*)\}$$

Eliminating p_i^* from both sides and multiplying by $z_i(p^*)$ we get

$$p_i^* z_i(p^*) \sum_j \max\{0, z_j(p^*)\} = z_i(p^*) \max\{0, z_i(p^*)\}$$

Summing up over i yields

$$\sum_i p_i^* z_i(p^*) \sum_j \max\{0, z_j(p^*)\} = \sum_i z_i(p^*) \max\{0, z_i(p^*)\}$$

The first summation on the LHS is zero, so

$$\sum_i [z_i(p^*) \max\{0, z_i(p^*)\}] = 0$$

Each bracketed term inside the summation is non-negative, since if $z_i(p^*) > 0$ then $\max\{0, z_i(p^*)\}$ is also strictly positive, and if
$z_i(p^*) \leq 0$ then $\max\{0, z_i(p^*)\} = 0$. But then if any of the $z_i(p^*) > 0$, $\sum_i [z_i(p^*) \max\{0, z_i(p^*)\}]$ could not be 0!

If any good is in strict excess supply in equilibrium then its price must be zero. If $p_i^* > 0$ and $z_i(p^*) < 0$ then $p_i^* z_i(p^*) < 0$. From Walras' law, $\sum_{j \neq i} p_j^* z_j(p^*) = -p_i^* z_i(p^*) > 0$. Since $\forall j$, $p_j^* \geq 0$ and $z_j(p^*) \leq 0$, $p_j^* z_j(p^*) \leq 0$, and therefore we have a contradiction. Hence $z_i(p^*) < 0$ implies $p_i^* = 0$.

0.2. Competitive Equilibrium. To proceed to a proof of the existence of Competitive Equilibrium, we must construct the excess demand function which results from *competitive* behavior and then show it satisfies the three assumptions stated in the first part. The following gives the flavor of this procedure leaving out some of the more troublesome details.

Let individuals $h = 1, ..., H$ with preferences $\succeq_h$ be supposed to hold endowment vectors e^h of goods $n = 1, ..., N$, and firms $m = 1, ..., M$ produce from production possibility sets Y^m.

At any non-negative N dimensional price vector p, firm m chooses a net output vector y^m that maximizes py^m over all the $y \in Y^m$ (i.e. firms profit maximize). Individual h, owns the fraction σ_{hm} of firm m and thus receives total income of

pe^h from endowments and of $\sum_m \sigma_{hm} py^m$ from profits in all of the firms. Individual h then chooses a bundle x^h which is most preferred from among all bundles which could be afforded, i.e.

$$px^h = pe^h + \sum_m \sigma_{hm} py^m$$

The total supply of goods in the economy is then given by the vector $\sum_h e^h + \sum_m y^m$ and the total vector of demands by $\sum_h x^h$, so excess demand is given by

$$z = \sum_h x^h - \sum_h e^h - \sum_m y^m$$

The profit maximizing net output vector for the firm is unchanged by a proportional change in prices. Then the individual income depends linearly on p and the individual budget set is homogeneous of degree zero in prices. That is, a proportional change in all prices leaves the individual's budget set unchanged. Thus the individual's preference maximizing bundle is also unchanged by a proportional change in prices. So z depends upon p and is homogeneous of degree zero in p.

Summing the individual budget constraints gives

$$\sum_h px^h = \sum_h pe^h + \sum_h \sum_m \sigma_{hm} py^m$$

or

$$p\left[\sum_h x^h - (\sum_h e^h + \sum_m y^m \sum_h \sigma_{hm})\right] = p\left[\sum_h x^h - (\sum_h e^h + \sum_m y^m)\right] = pz(p) = 0$$

which is simply Walras' law.

Continuity of $z(p)$ is more difficult to establish. In particular, it would be sufficient to show that the individual competitive demand and supply functions are continuous, i.e. $x^h(p)$ and $y^m(p)$. This is guaranteed if the indifference curves for individuals and production possibility sets for firms are strictly convex.