Lecture 5

Economic Analysis of Behavior under Risk

This chapter presents the economic modeling of decision-making under risk.

Choices under *risk* occur when the probability distribution of the outcomes are objectively known to the decision-maker. Choices under *uncertainty* occur when no objective probability distribution is given to the agent. This chapter focuses on decision-making under risk[1].

5.1 Introduction: an overview

The traditional approach to modeling behavior under risk is through the use of the **expected utility approach**. The expected utility model was suggested by Von-Neumann and Morgenstern in *The Theory of Games and Economic Behavior* (1944). Expected utility theory describes the relationship between an individual's scale of preferences for a set of acts and their associated consequences. Given certain postulates about rational choice, Von-Neumann and Morgenstern developed a set of axioms about the ordering, continuity, and independence of individual choice and used this as a base to derive the properties of the expected utility function, thus describing the conditions under which an individual's preferences under random choices correspond to maximization under the expected utility model.

Friedman and Savage's paper (1948) is the first where the expected utility approach is applied to explain economic behavior. The utility function is defined on wealth, where diversification and general risk aversion are explained by the function's concavity. Moreover, an S-shaped specification for the utility function ex-

[1]For a discussion on modeling choice under uncertainty, see Savage (1948).

plains why an individual may be risk averse for some choices but prefer risk for other choices.

Arrow (1971) and Pratt (1964) introduced measures of risk aversion. They defined:

$R_A = \Leftrightarrow \frac{U''}{U'}$, a measure of *absolute risk aversion* and

$R_R = \Leftrightarrow \frac{U''}{U'} W$, a measure of *relative risk aversion*,

where U'' and U' indicate, respectively, the second and first derivative of the von Neumann-Morgenstern utility function and W is wealth.

They established that, under the expected utility hypothesis, there exists a one-to-one relationship between preferences over random income (or wealth) and the measures of risk aversion. They claim that, as income grows, one cares less about one 'unit' of risk —the measure of absolute risk aversion is declining—, but cares equally about the risk involving a given share of his wealth —the measure of relative risk aversion may be constant and perhaps even equal to 1.

The next step in the theory of decision–making under risk was the development of models and concepts for measuring risk. The first efforts in this direction used statistical indexes as mean and variance of the random outcome as arguments of the utility functions. Recognizing that variance is not always a good measure of risk, Rothschild and Stiglitz (1970), Hancock and Levy (year?), and Radar and Russell (1969) developed models and concepts that are useful for a more general comparison of risky prospects. These approaches use probability distributions and are independent of the decision-maker's utility function. Concepts such as the *mean-preserving spread* and *stochastic dominance* fall under this rubric and have been used in much recent work.

In the following section we will discuss some of the main approches currently used to measure risk and their implications in terms of economic analysis.

5.2 Measures of Risk Aversion and Their Interpretation

What does it mean to say that an individual is risk averse in the context of expected utility? How can we measure people's attitudes towards risk? Kenneth Arrow (1971) developed answers to these questions. Working from the definition of a risk-averse individual as one who "*starting from a position of certainty, is unwilling to take a bet which is actuarially fair*", Arrow derives a series of quantitative measures of risk attitudes. The most important of these are the *absolute* and *relative risk-aversion* *coefficients*, which can be derived from the Von Neumann-Morgenstern utility function. Arrow also hypothesized that individuals would exhibit decreasing absolute risk aversion and increasing relative risk aversion under most circumstances. These hypotheses have critical implications for the empirical specification of utility functions. Some of the most simply formulated utility functions do not exhibit the risk preference structure postulated by Arrow.

5.2.1 Absolute Risk Aversion

Definition:

$$\text{Absolute risk aversion: } R_a = \Leftrightarrow \frac{U''(w)}{U'(w)}$$

where w is wealth, $U(\cdot)$ is the Von Neumann-Morgenstern utility function which is bounded and twice differentiable, U' is the marginal utility of wealth, and U'' is the rate of change of marginal utility with respect to wealth. The coefficient of absolute risk aversion directly measures how much over fair odds an individual requires before accepting a bet.

Absolute Risk Aversion with a Discrete Probability Distribution

Suppose a risk-averse individual is offered, for a small amount h, the following gamble:

win h with probability P or lose h with probability $1 \Leftrightarrow P$.

When will the individual be indifferent to the gamble? Suppose that the initial wealth level is w_0. Comparing the expected utility of taking the gamble with the utility achieved with no gamble we obtain:

$$PU(w_0 + h) + (1 \Leftrightarrow P)U(w_0 \Leftrightarrow h) = U(w_0) \tag{5.1}$$

Estimating a Taylor's series approximation around h:

$$U(w_0 + h) = U(w_0) + U'(w_0)h + \frac{1}{2}U''(w_0)h^2$$

$$U(w_0 \Leftrightarrow h) = U(w_0) \Leftrightarrow U'(w_0)h + \frac{1}{2}U''(w_0)h^2$$

Substituting these approximations into (5.1), we get:

$$PU(w_0 + h) + (1 \Leftrightarrow P)U(w_0 \Leftrightarrow h) \approx U(w_0) + (2P \Leftrightarrow 1)U'(w_0)h + \frac{1}{2}U''(w_0)h^2$$

Canceling like terms and rearranging terms, we obtain:

$$\begin{aligned}(2P \Leftrightarrow 1)U'(w_0)h &= \Leftrightarrow \frac{1}{2}U''(w_0)h^2 \\ (2P \Leftrightarrow 1) &= \frac{1}{2}R_A(w_0)h \\ P &= \frac{1}{2} + \frac{1}{4}R_A(w_0)h \\ \frac{dP}{dh} &= \frac{1}{4}R_A(w_0)\end{aligned}$$

Therefore, the coefficient of absolute risk aversion R_A tells how much the odds of winning have to be affected to induce a risk-averse individual to take a constant sum gamble. For a risk-averse individual, this coefficient should be positive. As the amount gambled increases, a higher probability of winning is needed in order for an individual to be indifferent between gambling and certainty.

Absolute Risk Aversion with a Continuous Probability Distribution

Consider a small gamble x with mean μ and variance σ^2. If we express this gamble in terms of the first two moments of the distribution and use a Taylor's series approximation around the mean, we get:

$$E\{U(w_0 + \mu) + U'(w_0 + \mu)(x \Leftrightarrow \mu) + \frac{1}{2}U''(W_0 + \mu)(x \Leftrightarrow \mu)^2\} \tag{5.2}$$

Defining a constant z as the certainty equivalent of X such that:

$$EU(w_0 + x) = U(w_0 + z)$$

and taking the expectations operator through (5.2), we obtain:

$$EU(w_0 + x) \equiv U(w_0 + \mu) + \frac{1}{2}U''(w_0 + \mu)\sigma^2 = U(w_0 + z)$$

Since both μ and z are small, we can make the following approximations:

$$U'(w_0) \approx U'(w_0 + \mu) \approx U'(w_0 + z)$$

and

$$U''(w_0) \approx U''(w_0 + \mu)$$

Therefore, we can approximate $U'(w_0)$ with the following equation:

$$U(w_0 + x) \approx U(w_0) + U'(w_0)\mu + \frac{1}{2}U''(w_0)\sigma^2 = U(w_0) + U'(w_0)z$$

Solving for z:

$$U'(w_0)\mu + \frac{1}{2}U''(w_0)\sigma^2 = U'(w_0)z$$

$$z = \mu + \frac{1}{2}R_A\sigma^2$$

Finally, expressing the risk-aversion coefficient in terms of the moments of the distribution:

$$R_A = \frac{\mu \Leftrightarrow z}{\frac{1}{2}\sigma^2}$$

For risk-averse individuals, μ is greater than z, so that R_A is positive.

5.2.2 Relative Risk Aversion

Definition

$$\text{Relative risk aversion: } R_R = \Leftrightarrow\frac{U''}{U'}w = R_A(w) \cdot w$$

Relative risk aversion is a measure of risk proportional to the level of wealth. It can be thought of as the elasticity of risk aversion. Consider a discrete probability distribution with a gamble to win a fraction of t of wealth with probability P or lose it with probability $1 \Leftrightarrow P$. When an individual is indifferent to the gamble,

$$PU[w(1+t)] + (1 \Leftrightarrow P)U[w(1 \Leftrightarrow t)] = U(w)$$

Following a similar procedure as with the absolute risk aversion coefficient, we can derive the following equation:

$$U(w) + (2P \Leftrightarrow 1)U'(w)tw + \frac{1}{2}U''(w)t^2w^2 = U(w)$$

Dividing through by $U\text{"}tw$ and canceling like terms, we obtain:

$$(2P \Leftrightarrow 1) = \Leftrightarrow\frac{1}{2}\frac{U''}{U'}tw = \frac{1}{2}R_Rt$$

The higher the relative risk aversion coefficient, R_R, the higher must be the probability of winning for the individual to be indifferent, for a given share of wealth at stake, t.

5.2.3 Hypotheses about Risk Preferences

In his 1971 essay, Ken Arrow put forward two hypotheses about the behavior of the measures of risk aversion. These are decreasing absolute risk aversion (DARA) and increasing relative risk aversion (IRRA). Decreasing absolute risk aversion implies that the willingness of individuals to take small bets of fixed size increases with wealth. Increasing relative risk aversion implies that, as wealth increases, the proportion of wealth the individual is willing to risk declines. An example of behavior which exhibits DARA and IRRA is as follows: Ross Perot will be more likely to accept a $10 bet than you or I (assuming no obscenely rich readers!), but he will be less likely to wager 10 percent of his wealth.

Going back to the derivations above, note that the assumption of *decreasing absolute risk aversion* is associated with:

$$\frac{dP}{dW} = \frac{1}{4}\frac{\partial R_A}{\partial W_0} < 0$$

From the derivation of the measure of relative risk aversion, we can obtain $\frac{\partial P}{\partial t} > 0$; and if we assume $\frac{\partial R_R}{\partial W} > 0$, then $\frac{\partial P}{\partial W} > 0$. We are likely to have an *increasing relative risk aversion* while we have *decreasing absolute risk aversion.*

5.2.4 The Implications of Risk Preferences for Empirical Specification of Utility Functions

Empirical applications of the expected utility framework may require specification of the utility function. In order to achieve a reasonable depiction of reality with a tractable form, it is desirable that the specific utility function $U(W)$, where W is wealth (or income), will have some of the following characteristics:

1. Simplicity.

2. Positive and decreasing marginal utility ($U' > 0$, $U'' < 0$).

3. Decreasing (or at least non–increasing) absolute risk aversion ($\frac{\partial R_A}{\partial W} \leq 0$, where $R_A = \Leftrightarrow\frac{U''(W)}{U'(W)}$)

4. Non-decreasing relative risk aversion ($\frac{\partial R_R}{\partial W} \geq 0$, where $R_R = \Leftrightarrow\frac{U''(W)W}{U'(W)}$). If R_R is constant, it is preferably near 1.

Specific Cases

- Quadratic Utility Functions: $U(W) = a + bW \Leftrightarrow \frac{1}{2}CW^2$, when $W > \frac{b}{c}$
  This utility function may be objectionable because it implies increasing absolute risk aversion $[R_A = c/(b \Leftrightarrow cW), \frac{\partial R_A}{\partial W} = cR_A/(b \Leftrightarrow cW)]$.

- Cobb-Douglas Utility Functions: $U(W) = AW^{\alpha}, \quad 0 < \alpha < 1$
  This utility implies constant relative risk aversion and decreasing absolute. risk aversion with $R_R(W) = 1 \Leftrightarrow a$. The deficiencies of this function are:

  1. $R_R = 1$ when $\alpha = 0$, and
  2. the expectation of utility under this function, $E(AW^{\alpha})$, may result in complex expressions.

- Logarithmic Utility Function: $U(W) = \ln W$
  This function implies constant relative risk aversion with $R_R(W) = 1$ and decreasing absolute risk aversion. This function is defined only for $W > 0$. The expected value of this utility, $E(\ln W)$, may be cumbersome in problems where W is a linear function of decision variables.

- Exponential Utility Functions: $U(W) = 1 \Leftrightarrow e^{-rW}$
  This utility function implies constant absolute risk aversion with $R_A(W) = r$. This functional form is easy to apply with distributions which can be defined by their moment-generating functions. Moment-generating functions are a function of the parameters of the distribution associated with random variables For example, for the exponential utility function, $EU(W) = 1 \Leftrightarrow E[e^{-rW}]$ and $E[e^{-rW}]$ is a moment-generating function.

For examle, let z be a random variable. The moment-generating function for z is $M_z(t) = E(e^{zt})$. When W is a normally distributed random variable with $E(W) = \mu$, $\mathrm{var}(W) = \sigma^2$, the moment generating function to the second order gives:

$$EU(W) = 1 \Leftrightarrow e^{-r\left[\mu - \frac{1}{2}r\sigma^2\right]}$$

Since any solution that maximizes $\mu \Leftrightarrow (1/2)r\sigma^2$ also maximizes $EU(W)$ in this case, when utility is exponential and the random variable is normally distributed, *maximization of a linear function of the mean and variance of income is equivalent to expected utility maximization*, i.e.,

$$\max_X EU(W) = \max_X 1 \Leftrightarrow e^{-rW}$$

Let

$$W = W_0 + \bar{P}X\varepsilon \Leftrightarrow C(X)$$

where:

X = output

P = average price

ε = random price variability, $\varepsilon \sim N(1, \sigma^2)$

$C(X)$ = cost function, $C' > 0$, $C'' > 0$

Expected utility maximizing outcome in this case can be obtained solving

$$\max_X \bar{P}X \Leftrightarrow C(X) \Leftrightarrow \frac{1}{2} r\sigma^2 \bar{P}^2 X^2$$

The optimal solution in this case is obtained solving the FOC:

$$\bar{P} \Leftrightarrow C'(X) \Leftrightarrow r\sigma^2 \bar{P}^2 X = O$$

Suppose $C(X) = cX$. The first-order condition becomes

$$\bar{P} \Leftrightarrow c \Leftrightarrow r\sigma^2 \bar{P}^2 X = 0$$

$$X = \frac{\bar{P} \Leftrightarrow c}{r\sigma^2 \bar{P}^2}$$

Questions:

1. Can supply (X as a function of $\bar{P}$) be negatively sloped in this case? Why?

2. Derive supply when price P follows a gamma distribution (see Yassour, Rausser and Zilberman)

5.2.5 Estimation of Risk Aversion Coefficient

The Arrow-Pratt measures of risk aversion and conceptual models such as Sandmo's have established a very rich theory of decision making under uncertainty in agricultural production and resource use. However, little empirical work has been done to test the theory. One of the challenges of applied econornics and agricultural economics research is to develop such an empirical base. This section will identify some of the problems and alternative approaches to address them.

The most difficult problem in assessing empirical expected utility models is the unobservable nature of an individual's evaluation of utility levels and the probabilities associated with them. Expected utility models assume that choices under uncertainty result from a mental process where the utility of many wealth levels are assessed and multiplied with the right probabilities —resulting in the expected utility of each prospect. Expected utilities are then compared to obtain the optimal choice. Unfortunately, these evaluations of utility levels are not observable. However, for standard economic analysis, we have to rely on observable variables. These include choices, for example, to adopt or not to adopt a technology, as well as characteristics of decision makers such as farm size, age, education, etc. Information about the subjective probabilities of different outcomes is also not available, which makes the estimation of decision makers' parameters even more difficult.

One approach for empirical estimation of expected utility is to reduce the uncertainty of the researcher by conducting experiments where the decision maker is presented with rewards and the probabilities of rewards is specified. She has to make choices between outcomes. In these cases, the researcher does not know the utility functions but supposedly knows everything else associated with the decision. Such experiments can be used to assess empirically to what extent the expected utility model is realistic, and in the case where it is a realistic description of reality, what the values of key parameters are —such as the measures of absolute and relative risk aversion.

In later lectures we will discuss some of the experiments and tests of expected utility hypothesis. In this lecture we will concentrate on models that try to estimate risk aversion parameters and, in particular, to test whether there is decreasing absolute risk aversion, increasing relative risk aversion, and if relative risk aversion is around one.

Binswanger used experiments where farmers in India were given actual monetary rewards as part of different gambles. Binswanger recorded their individual choices and elicited from them some of their risk–aversion parameters. Since the income level of the individuals involved was relatively low, with a reasonable amount of money, he was able to collect a large set of data. He found substantial variability in the measures of risk aversions between individuals, who demonstrate heterogeneity in risk preferences. He also found that, for any given individual the measure of partial relative risk aversion did not change much with different gambles.

Other experimental studies, especially ones conducted in the United States, were not done with actual rewards but instead with hypothetical choices. These studies were mostly intended to identify paradoxes in risk choice patterns that contradict expected utility and not to elicit the parameters of the utility function.

The most popular approaches to estimate the parameters of the utility function were ones using programming models and econometric models. The following section describes how one can use econometric models such as the Just-Zilberman model we mentioned in the last lecture to elicit risk aversion coefficients.

Estimation

Consider from simple formulations of the model the case where the farmers have to allocate land between two crops. It is assumed that the production function has fixed proportions per acre. Therefore, the farmer has a mean variance and covariance between the profits of the crops per acre. As the analysis in the previous section suggests, the amount of land allocated for tcchnology 1, L_1 can be written as:

$$L_1 = \frac{\mu_1 \Leftrightarrow \mu_2}{(\sigma_1^2 + \sigma_2^2 \Leftrightarrow 2\sigma_{12})\phi(\bar{W})} + \frac{\sigma_2 \Leftrightarrow \sigma_{12}}{\sigma_1^2 + \sigma_2^2 \Leftrightarrow 2\sigma_{12})}\bar{L} \tag{5.3}$$

where μ_1 is mean profit per acre of technology 1, μ_2 is mean per acre of profit of technology 2, σ_1^2 is the variance of profit per acre of technology 1, σ_2^2 is the variance of profit per acre of technology 2, and σ_{12} is the covariance between profits per acre. The function $\phi(\bar{W})$ is a measure of absolute risk aversion as a function of average wealth, $\bar{W}$.

This model, which is based on a Taylor series as an approximation of the first-order condition, allows us to get a quantitative assessment of the behavior of measure of risk aversion. Since one cannot observe the utility function change with wealth for one individual, in this model we try to estimate how the measure of risk aversion is changing between individuals where we tried to estimate the measure of risk aversion as a function of average wealth, $|barW$, using our results derived by our approximation.

Several relationships which can be derived using these models depend on the data one has and the degree of ease or statistical sophistication.

Now consider the most simplistic case. One assumes constant absolute risk aversion and has only data on acre and land allocation with two crops. In this case, the estimated models will be $L_1 = A_1 + A_2\bar{L} + \varepsilon$. In this case, ε is a random variabile. The estimated value of $\hat{A}_1$ is an estimator of A. The estimated value of $\hat{A}_2$, is an estimator of B.

Such model can be estimated using a simple linear regression. One can use the model to test simple empirical hypothesis. For example, if crop 1 has a higher mean and higher variance, and the correlation of yield between the two crops is not very big, one can test hypotheses that A_1 is positive and A_2 is negative and smaller than 1. If one ran such models in different regions and obtained a B_1 estimate for the two regions, then one may test an hypothesis that has a region where the correlation between yield is bigger and may have a smaller B_1.

One can use this land allocation equation and incorporate it with other elements that determine land allocation, for example, fixed costs of different technologies as well as credit constraints. In this case one can get the systems, and several segments and risk aversion will determine the relation between land allocation to crop 1 and total land only in one segment. Marra and Carlson have a nice application of this approach to allocate assessment of adoption of double cropping in the United States.

Data on L_1 and L can lead to more insightful results when a more sophisticated econometric model is used and added assumption on risk aversion is introduced if, instead of assuming constant absolute risk aversion one assumes that ϕ varies with $\bar{W}$. Furthermore, if ones assumes a constant ratio between farm size and expected wealth $\bar{W} \approx \alpha \bar{L}$, then risk aversion can be approximated and estimated as a function of farm size. Suppose

$$\phi(\bar{W}) = CW^{-\eta}$$

where η is the elasticity of absolute risk aversion

$$\eta = \Leftrightarrow \frac{\partial \phi}{\partial \bar{W}} \frac{\bar{W}}{\phi}$$

and C is a scaling constant. In the case of constant absolute risk aversion, $\eta = 0$. When we have decreasing absolute risk aversion, $\eta > 0$. Let $r = \phi W$ be a measure of relative risk aversion. Note that if we define the elasticity of relative risk aversion as δ, it is:

$$\delta = \frac{\partial r}{\partial \bar{W}} \frac{\bar{W}}{r} = \left[\frac{\partial \phi}{\partial \bar{W}} \frac{\bar{W}}{\phi} + 1 \right] = 1 \Leftrightarrow \eta$$

So, in the case of constant relative risk aversion, $\eta = 1$ and in the case of increasing relative risk aversion, $\eta < 1$. Thus, assuming decreasing absolute and increasing relative risk aversion implies $0 < \eta < 1$ (or $0 < \delta < 1$).

Thus, when $\phi(\bar{W}) = C\bar{W}^{-\eta}$ and $\bar{W} = \alpha \bar{L}$, the absolute risk aversion coefficient, ϕ is $\phi = C\alpha^{-\eta} L^{-\eta}$ and the estimatable model that corresponds to these assumption is

$$L_1 = A_3/L^{-\eta} + A_2 \bar{L} = A_3 L^{\eta} + A_2 \bar{L}$$

when $A_3 = A_1 / C\alpha^{-\eta}$.

An alternative formulation is

$$\frac{L_1}{L_2} = A_2 + A_3 L^{-\delta}$$

Estimation of these models provides $\hat{A}_2$, $\hat{A}_3$, and $\hat{\delta}$ or $\hat{\eta}$. That allows testing when absolute risk aversion is decreasing in wealth (approximated by size) and relative risk aversion is increasing in wealth.

This formulation can be expended to identify other factors affecting risk aversion. If socioeconomic data are available (age, education, etc.), one may replace C in the expression for ϕ with $g(S)$, a function of socioeconomic variable. One plausible specification is $\phi(\bar{W}) = g(S)\bar{W}^{-\eta}$. This specification can lead to an estimatable relationship which is a function of farm size and socioeconomic variables. More detailed data on profit and wealth may allow all estimation of the Just and Zilberman model with less approximation. Even then, one may need to use heroic assumptions. Data on the subjective values of mean at the individual farm levels are not easily obtainable. Thus, one may need to estimate these variables as well and introduce them to an estimatable form. Similarly, average wealth of individual farmers are needed to be computed from other accounting data. In essence, estimation of risk aversion coefficient from a simple specification such as (5.3) requires much compromise and ingenuity. We also have to recognize that the decision maker has the same problem of data assembly as the researcher. None of the farmers knows his μ_1, μ_2 and σ_2. Even if she follows something resembling the expected utility criteria, she has to estimate key parameters. Thus, in essence, a more complete model should recognize that.

Collender, Chalfant, and Subramanian developed an approach when risk-aversion parameters are utilized by farmers who recognize the uncertainty of his estimate of the key parameters of his profit distribution. The model is complex, but the optimal L_1 depends not only on the estimated means of variance of profits and the measures of risk aversion but also on measures of the estimators' reliability and the moments of the farmers' profit.

5.3 The Use of Expected Utility for Understanding Producer Behavior

5.4 Uncertainty in Production

5.4.1 Introduction

Uncertainty occurs in several aspects of the production process. Various institutional mechanisms for reducing these uncertainties have been developed. A partial list of these and the type of production uncertainty they address is as follows:

Types of Risks Faced by Producers	Mechanisms to Remedy Them
Output price risk	Future markets, Forward contracts
Yield risk	Crop insurance
Labor supply availability	Mechanization, Long-term labor contract
Input reliability	Product warranty
Input price uncertainty	Forward contacts
Government policy uncertainty	

Many other institutions were developed to address risks faced by firms. In this section we will focus on the building blocks of modeling production uncertainty

5.4.2 Risk Specification

Production risk is generally modeled through alternative specifications of the production function. Several examples of production functions and their implications for risk analysis are discussed below.

Let Y be output quantity, X input quantity and ε a random variable in each of the following models.

Model 1. Additive risk

$$Y = f(X) + \varepsilon, \qquad E(\varepsilon) = 0$$

In this case input use does not affect risk and the only type of risk considered is output risk.

Model 2. Multiplicative risk

$$Y = f(x) \cdot \varepsilon, \qquad E(\varepsilon) = 1$$

In this case any input that increases mean yield also increases risks associated with yields.

Model 3. Linear risk (Just and Pope production function)

$$Y = f(X) + g(X)\varepsilon, \qquad E(\varepsilon) = 0$$

Under this production function specification, impacts of inputs on yield and risk can be differentiated. For example, some inputs may be yield increasing $f' > 0$, and risk reducing $g' < 0$ and others may increase both yield and risk $f' > 0$, $g' > 0$.

5.4.3 The Sandmo-Leland Model

How will a firm behave when output price is a random variable as opposed to when it is a constant average?

5.4.4 Sandmo's Model

Sandmo (1971) developed a model of a competitive firm facing price uncertainty. He uses a multiplicative risk specification. In his model, the only risk producers face is output price risk; thus P is a random variable with mean P and $E(P) = \bar{P}$

Firms maximize expected utility with cost function $C(Y)$ where $C' > 0$, $C'' > 0$. The decision problem is

$$L = max_Y EU(PY \Leftrightarrow C(Y) + W_0)$$

The first-order condition is

$$\frac{\partial L}{\partial Y} = EU'(PY \Leftrightarrow C(Y) + W_0)[P \Leftrightarrow C'(Y)] = 0 \tag{5.4}$$

By using the statistical theorem concerning the expected value of the product of two random variables which states:

$$E(XZ) = E(X)E(Z) + \mathrm{Cov}(XZ)$$

we can rewrite (5.4)as:

$$[P \Leftrightarrow C'(Y)] + \frac{Cov[U'(W), P \Leftrightarrow C'(Y)]}{EU'(W)} = 0$$

where $W = PY \Leftrightarrow C(Y) + W_0$.

In order to determine the impact of risk on output, we need to determine the signs of each element in this expression. We will make use of the fact that

$$Cov[U'(W), P \Leftrightarrow C(Y)] = E[U'(W)(P \Leftrightarrow C'(Y))]$$

to find the sign of the latter.

1. By definition, $W = PX \Leftrightarrow C(Y) \Leftrightarrow W_0$ and $E(W) = \bar{P}Y \Leftrightarrow C(Y) \Leftrightarrow W_0$. Therefore,

$$W \Leftrightarrow E(W) = (P \Leftrightarrow \bar{P})Y$$

$$\therefore W = E(W) + (P \Leftrightarrow \bar{P})Y$$

2. If we assume $P > \bar{P}$, from step (1) we know that W will he greater than $E(W)$. Therefore,

$$U'(W) \leq U'[E(W)]$$

3. If we multiply $P \Leftrightarrow \bar{P})$ through the results from (2), and take the expectation of both sides, we get:

$$U'(W)(P \Leftrightarrow \bar{P}) \leq U'[E(W)]E(P \Leftrightarrow \bar{P})$$

Since $E(P \Leftrightarrow \bar{P}) = 0$, $U'(W)(P \Leftrightarrow \bar{P})$ is less than zero.

Under risk neutrality, $\bar{P} = C'$, but with risk–averse behavior $\bar{P} \Leftrightarrow C(X) > 0$, which implies $\bar{P} > C(X)$. Risk-averse firms will produce less than risk-neutral firms. This finding implies that price stabilization policies will lead to an increase in output.

(figure risk4 here)

The results of the Sandmo model can be contrasted with results from the stabilization literature. With stabilization models, producers can adjust to changes in an unstable situation —which could lead to higher output under uncertainty. Under the assumptions of the Sandmo model, this is not possible.

Suppose utility depends on wealth and $W = PY \Leftrightarrow C(Y) + W_0$. What will be the impact of higher wealth on Y?

Returning to the FOC of the firm's profit maximization:

$$\frac{\partial L}{\partial Y} = EU'(PY \Leftrightarrow C(Y) + W_0)[P \Leftrightarrow C'(Y)] = 0$$

the second-order condition is

$$\frac{\partial^2 L}{\partial Y^2} = EU''(P \Leftrightarrow C(Y))^2 \Leftrightarrow EU'C'' < 0$$

Total differentiation of the first-order condition with respect to W_0 yields:

$$\frac{dY}{dW_0} = \Leftrightarrow \frac{EU''(P \Leftrightarrow C(Y))}{\frac{\partial^2 L}{\partial Y^2}} \tag{5.5}$$

Under the assumption of decreasing absolute risk aversion, we have:

$$\Leftrightarrow \frac{U''}{U'} < R_A(\tilde{W})$$

with $P < \tilde{P}$ and $W > \tilde{W}$; $\tilde{W}$ being the level of wealth associated with $\tilde{P}$, and $\tilde{P} = C'(Y)$. Therefore,

$$\Leftrightarrow \frac{U''}{U'}[P \Leftrightarrow C'(Y)] < R_A(\tilde{W})[P \Leftrightarrow C'(Y)]$$

for every W. From this equation, we can derive:

$$\Leftrightarrow U''[P \Leftrightarrow C'(Y)] < R_A(\tilde{W})U'(P \Leftrightarrow C'(Y))$$

We know from the firm's first-order condition that the right–hand side should equal zero at an optimum. Therefore, the left–hand side of the equation is less than zero. From these results, we can sign $\frac{\partial Y}{\partial W_0}$.

Going back to equation (2), we can sign each of the elements of the equation as:

$$\Leftrightarrow\frac{(+)}{(\Leftrightarrow)}$$

which is positive.

More affluent risk-averse producers will provide more output, and we found a positive wealth effect on supply. To summarize, we found that: $\frac{\partial Y}{\partial W_0} \gtreqless 0$ if $\frac{\partial R_A}{\partial W_0} \lesseqgtr 0$ which means that risk–neutral firms will produce more than risk–averse ones, and, under decreasing absolute risk aversion, wealthier producers will produce more.

Sandmo also shows that increasing relative risk aversion may lead to a reduction in supply as income taxes increase, and he was able to partially show that increase in riskiness (measured by a mean-preserving spread) reduces supply. His technique was used to obtain conceptual results in many problems with multiplicative risk.

Let $P_1 = \gamma P + (l \Leftrightarrow \gamma)\bar{P}$. Using this in the firm's profit maximization, we obtain:

$$L = \max_X EU(P_1 Y \Leftrightarrow C(Y) + W_0)$$

$$P_1 = \gamma P + (l \Leftrightarrow \gamma)\bar{P}$$

$$\frac{\partial Y}{\partial \gamma} < 0 \qquad \text{if } \gamma = 1$$

These results indicate that there will be two supply effets: One from price and another from uncertainty.

Sandmo's approach, while general, cannot be applied to situations with multiple and correlated risks. One may need to specify utility functions in more detail to address these problems.

Summary

The model results indicated that:

1. Under uncertainty, less output will be produced.

2. An increase in risk aversion will reduce output.

3. Optimal resource allocation by a risk-averse firm requires that the value of the marginal product of a resource exceeds its rental value.

4. If there is decreasing absolute risk aversion, reductions in fixed cost will increase output. Thus, financial conditions may affect production decisions

5. Expected profit will be highest for firms which are closest to being risk neutral and have the highest output.

Sandmo's technique was extended by Feder (1979) to model situations where multiplicative risk arises, in particular, the adoption of new technologies in less-developed countries. The Feder model is formulated as:

$$\max EU(g(y) + f(y) \cdot \varepsilon)$$

The Sandmo approach was used by Batra (1974) for input demand; by Feder for technological adoption (1980); and by Feder, Just, and Schmitz (?) for futures market behavior. The limitations of this model are: (i) the multiplicative assumption does not allow for the analysis of the differential impacts of an input on output vs. production risk, and (ii) it is only possible to handle one random variable at a time. The advantage of the model is its generalized form, allowing for the use of loss restrictive utility forms, in contrast with specific functional forms.

5.4.5 The Mean–Variance Approach

This approach was introduced in works by Tobin (1959) and by Markovitz (1959). It plays a key role in finance —being used as a base for capital asset pricing. The basic idea of this model is that utility from random prospects can be described as a function of the moments of the distribution around a mean outcome $\bar{y}$ through a Taylor's series expansion:

$$EU(y) = \int U(y) f(y) dy$$

$$U(y) = U(\bar{y}) + U'(\bar{y})(y \Leftrightarrow \bar{y}) + U''(\bar{y}) \frac{(y \Leftrightarrow \bar{y})^2}{2} + \sum_{n=3}^{\infty} U^n \frac{(y \Leftrightarrow \bar{y})^n}{n!}$$

from which, taking the expectations of both sides:

$$EU(y) = U(\bar{y}) + \sum_{n=2}^{\infty} \frac{U^n(\bar{y})}{n!} E(y \Leftrightarrow \bar{y})^n = f(M_1, M_2, M_3, \dots)$$

If a distribution can be completely defined by n moments, then the expected utility is a function of these moments.

If a distribution is defined by its first two moments, then the expected utility is a function of the distribution's mean and variance. For example, in the case of financial assets, the price of any asset is determined by its mean return and its variance with the market portfolio. Certain restrictive conditions on the utility functions and the distribution of the random outcome variable are required in order to be able to express expected utility as a function of the mean and variance. These are: (i) the utility function must be quadratic or exponential in form and (ii) the outcome variable should be normally distributed. Given these conditions, the expected utility can be expressed as:

$$EU(y) = \alpha \bar{y} + \beta \frac{\sigma^2}{2}$$

Freund (1956) proved the linearity of the expected utility function under the condition of normality and exponential utility. The linear mean variance approach has one big advantage —it is easy to work with and it allows the consideration of behavior under risk with a large number of random variables. It is used very frequently (see Just and Zilberman, 1983). However, the model is objectionable on three grounds:

1. Quadratic utility implies increasing absolute risk aversion.
2. Exponential utility implies constant absolute risk aversion.
3. Normality of the outcome variable may be unreasonable (e.g., crop yields have a negative gamma distribution as Day (1965) has shown).

Despite these shortcomings, the linear mean variance approach is popular since it results in models useful for dynamic programming. In the next section some examples of how this approach has been applied are discussed.

Applications of the Mean-Variance Approach

1. Linear and Quadratic Programming. — The mean–variance approach can be very useful in modeling a typical farmer's land allocation problem. Assuming linear solution techniques, the farmer's problem can be stated as (using matrix notation):

$$\max U'L \Leftrightarrow rL'\Sigma L \Leftrightarrow V'L$$

where:

U = average yield vector

V = variable cost vector

$\Sigma =$ variance-covariance matrix of revenue per acre, and

$r =$measure of risk.

One way to apply the mean variance approach is to construct an efficiency locus of mean variance (or standard deviation) trade–offs. This is done through a quadratic programming problem where the land allocation that minimizes variance is computed to attain expected profit from the land, $\bar{Z}$.

$$\min_L L'\Sigma L$$

subject to a mean income constraint:

$$L'U = \bar{Z}$$

2. MOTAD. — The quadratic programming procedure is computationally expensive. A simpler procedure is to calculate the minimum of total absolute deviation (MOTAD). Assuming that there are J crops and H states of the world, the problem is set up as follows.

First, an average return to each crop over all states, $\bar{U}_j$, is calculated, given

L_j= land allocated to crop j,

U_{jh} = revenue per acre of crop j in state of world h

$$\bar{U}_j = \frac{1}{n} L_{jh}$$

To calculate MOTAD, we use the formula:

$$\min \sum_{h=1}^{H} y_h$$

where y_h is the absolute total deviation in state h, subject to:

$$\sum_{j=1}^{J} (U_{jh} \Leftrightarrow \bar{U}_j) L_j + y_h \geq 0$$

$$\sum_{j=1}^{J} \bar{U}_j L_j \leq \bar{Z}$$

$$\sum_{j=1}^{J} a_{ij} \leq b_i$$

where the $b_i's$ are other linear programming constraints and the $a_{ij}'s$ are other linear programming coefficients. The MOTAD procedure generates a mean–absolute deviation trade–off. It is a reasonable approximation to the more costly to calculate mean standard error trade–off.

Mean–Variance models

When the distribution of wealth (or profit) has two parameters, expected utility can be expressed as function of the mean and variance of wealth (profit). With the negative exponential utility function and normal distribution, utility maximization can be expressed as a linear function of mean and variance of wealth (profit). These assumptions have been used extensively, especially in agricultural economics and in finance. They are used primarily in conceptual analysis when decision makers are affected by several correlated random variables when more general frameworks, such as Sandmo's, cannot be easily applied.

The linear mean-variance formulation has been extensively used in modeling land allocation by farmers. Assume that a farmer has L acres of land which can be divided among n crops. The profit of the ith crop is normally distributed with mean μ_i, variance σ^2, and $cov(\pi_i, \pi_j) = \sigma_{ij}$. The land allocation problem is a constrained quadratic programming problem

$$\max_{L_i} \sum_{i=1}^{N} L_i \mu_i \Leftrightarrow \frac{r}{2} \left[\sum_{i=1}^{N} \left(L_i^2 \sigma_i^2 \right) + \sum_{j \neq i} L_i L_j \sigma_{ij} \right]$$

s.t.

$$\sum_{i=1}^{n} L_i \leq \bar{L}$$

Let λ be the shadow price of land. The Lagrangian formulation of this problem is:

$$L = \max_{L_i} \sum_{i=1}^{N} L_i \mu_i \Leftrightarrow \frac{r}{2} \left[\sum_{i=1}^{N} \left(L_i^2 \sigma_i^2 \right) + \sum_{j \neq i} L_i L_j \sigma_{ij} \right] + \lambda \left[\bar{L} \Leftrightarrow \sum_{i=1}^{N} L_i \right]$$

The optimality conditions, when there is an internal solution, are:

$$\frac{\partial L}{\partial L_i} = \mu_i \Leftrightarrow r \left[L_i \sigma_i^2 + \frac{1}{2} \sum_{j \neq i} L_j \sigma_{ij} \right] \Leftrightarrow \lambda = 0, \qquad i = 1, \ldots, N \tag{5.6}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{N} L_i \leq \bar{L} = 0$$

At the optimal solution, λ equals the marginal contribution of land to expected net benefits. For the ith crop, this marginal value is equal to net profit per acres, μ_i, minus marginal contribution to risk (MV_i, where MV_i = risk cost). Note that the marginal contribution to overall risk depends on the inherent risk of the crop and correlation of its profits with the profits from the other crops. A crop for which profits are negatively correlated to the profits of other crops may reduce the overall cost of risk. Such a crop may have substantial acreage even if it is less profitable on average than other crops. By averaging all the first-order conditions, the marginal value of land λ can be expressed as

$$\lambda = \bar{\mu} \Leftrightarrow \bar{M}V$$

where:

$$\bar{\mu} = \frac{1}{N} \sum_{i=1}^{N} \mu_i$$

and

$$\bar{M}V = \frac{1}{N} \sum_{i=1}^{N} MV_i$$

The first-order condition (5.6) can be rewritten as:

$$\mu_i \Leftrightarrow \bar{\mu} \Leftrightarrow (MV_i \Leftrightarrow \bar{M}V) = 0$$

A crop will be grown if it is either more profitable or less risky than average. Riskiness may not reflect less variability but, rather, negative correlation of profit with the other crops.

Linking Mean–Variance and General EU Models

There have been many attempts to generalize the linear mean–variance framework. One such attempt is presented in Just and Zilberman (*Oxford Economic Papers, 1983*).

Consider the case when two crops are grown and each has a costant return–to–scale technology when profits per acre, Π_i, is a random variable with mean μ_i. When all land is used, the expected utility problem bocomes

$$\max_{L_i} EU[\Pi_1 L_1 + \Pi_2(\bar{L} \Leftrightarrow L_1) + W_0] \tag{5.7}$$

and the first-order condition is

$$E\{U'(w)(\Pi_1 \Leftrightarrow \Pi_2)\} = 0 \tag{5.8}$$

Marginal utility at W can be approximated by

$$U'(W) = U'(\bar{W}) + U''(W)\left[L_1(\Pi_1 \Leftrightarrow \mu_1) + (\bar{L} \Leftrightarrow L_1)(\Pi_2 \Leftrightarrow \mu_2)\right] \tag{5.9}$$

where $\bar{W} = W_0 + \mu_1 L_1 + \mu_2(\bar{L} \Leftrightarrow L_1)$. By introducing this approxirnation to the first-order condition, (5.8) becomes

$$\begin{aligned} &E\left\{U'(\bar{W})(\Pi_1 \Leftrightarrow \Pi_2) + U''(\bar{W})\left[L_1(\Pi_1 \Leftrightarrow \mu_1) + (\bar{L} \Leftrightarrow L_1)(\Pi_2 \Leftrightarrow \mu_2)\right](\Pi_1 \Leftrightarrow \Pi_2)\right\} = \\ &= U'(\bar{W})\left\{\mu_1 \Leftrightarrow \mu_2 \Leftrightarrow R_A(W)\left[L_1(\sigma_1^2 \Leftrightarrow \sigma_{12}) + (\bar{L} \Leftrightarrow L_1)(\sigma_{12} \Leftrightarrow \sigma_2^2)\right]\right\} = 0 \end{aligned} \tag{5.10}$$

The first-order condition suggests that

$$L1 = \frac{\mu_1 \Leftrightarrow \mu_2}{V(\Pi_1 \Leftrightarrow \Pi_2) R_A(\bar{W})} + \frac{\sigma_2^2 \Leftrightarrow \sigma_{12}}{V(\Pi_1 \Leftrightarrow \Pi_2)}\bar{L} \tag{5.11}$$

where $V(\Pi_1 \Leftrightarrow \Pi_2) = \sigma_1^2 + \sigma_2^2 \Leftrightarrow 2\sigma_{12}$.

Under risk neutrality, a producer will specialize in the crop with higher mean profit. With risk aversion, consideration of riskiness is added to those of average profitability, and the weight of the expected profit differential in determining L_1 declines as $V(\Pi_1 \Leftrightarrow \Pi_2)$ increases. Suppose crop 1 has a higher profit and higher risk than crop 2. For example, it maybe a modern export crop that is sensitive to weather and economic conditions, while crop 2 may be a traditional crop. It may be of interest to understand the impact of farm size on the acreage of each crop. Differentiation of (5.11) yields

$$\frac{dL_1}{d\bar{L}} = \Leftrightarrow \frac{\mu_1 \Leftrightarrow \mu_2}{V(\Pi_1 \Leftrightarrow \Pi_2) R_A(\bar{W})} \cdot \frac{dR_A}{d\bar{W}} \cdot \frac{\bar{W}}{R_A} \cdot \frac{d\bar{W}}{d\bar{L}} \cdot \frac{1}{\bar{W}} + \frac{\sigma_2^2 \Leftrightarrow \sigma_{12}}{V(\Pi_1 \Leftrightarrow \Pi_2)} \tag{5.12}$$

Substituting (5.11) into the equation gives us

$$\begin{aligned} \frac{dL_1}{d\bar{L}} &= \left[\frac{\sigma_2^2 - \sigma_{12}}{V(\Pi_1 - \Pi_2)}\bar{L} \Leftrightarrow L_1\right] \frac{dR_A}{d\bar{W}} \cdot \frac{\bar{W}}{R_A} \cdot \frac{d\bar{W}}{d\bar{L}} \cdot \frac{1}{\bar{W}} + \frac{\sigma_2^2 - \sigma_{12}}{V(\Pi_1 - \Pi_2)} = \\ &= \frac{\sigma_2^2 - \sigma_{12}}{V(\Pi_1 - \Pi_2)}\left[1 + \bar{L}\frac{dR_A}{d\bar{W}} \cdot \frac{\bar{W}}{R_A} \cdot \frac{d\bar{W}}{d\bar{L}} \cdot \frac{1}{\bar{W}}\right] \Leftrightarrow L_1 \frac{dR_A}{d\bar{W}} \cdot \frac{\bar{W}}{R_A} \cdot \frac{d\bar{W}}{d\bar{L}} \cdot \frac{1}{\bar{W}} \end{aligned} \tag{5.13}$$

Let $\eta_R = \Leftrightarrow \frac{dR_A}{d\bar{W}} \frac{\bar{W}}{R_A}$ be the elasticity of absolute risk aversion, and assume increasing absolute risk aversion, $\eta_R(W) > 0$. With this definition, the change in crop 1's acreage with respect to size can be presented as:

$$\frac{dL_1}{d\bar{L}} = \frac{\sigma_2^2 \Leftrightarrow \sigma_{12}}{V(\Pi_1 \Leftrightarrow \Pi_2)}\left[1 \Leftrightarrow \eta_R \frac{\bar{L}}{\bar{W}} \frac{d\bar{W}}{d\bar{L}}\right] + \eta_R \frac{L_1}{\bar{L}} \frac{\bar{L}}{\bar{W}} \frac{d\bar{W}}{d\bar{L}}$$

It is reasonable to argue that (i) $0 < \eta_R < 1$ when relative risk aversion is increasing and (ii) $(d\bar{W}/d\bar{L}) \cdot (\bar{L}/\bar{W})$ is also likely to be smaller than 1. If we examine the case of constant absolute risk avetsion where $\eta_R = 0$, then

$$sign\left(\frac{dL_1}{d\bar{L}}\right) = sign\left(\frac{\sigma_2^2 \Leftrightarrow \sigma_{12}}{V(\Pi_1 \Leftrightarrow \Pi_2)}\right) \tag{5.14}$$

Note that $\sigma_2^2 \Leftrightarrow \sigma_{12} = \sigma_2[\sigma_2 \Leftrightarrow \rho\sigma_1]$, where ρ is the correlation between the crop i's profits and the σ_i's are the standard deviation of crop i's profits. Equation (5.14) suggests that, if crop 2 is much less risky than crop 1 and the correlation between their profit is high (so that $\sigma_2 < \rho\sigma_1$), risk consideration will cause larger farmers to grow less of the risky crop than the smaller farmers. In other cases, Just and Zilberman have shown that, with $0 < \eta_R < 1$, the land share of the more risky technology *declines* with farm size. Since risk costs increase more than proportionally wich farm size, *larger formers* are likely to grow *relatively* less of the risky crops. The optimization problem (5.7) is not a constrained one, but we have to realize that $0 \leq L_1 \leq \bar{L}$. Therefore, letting the result of this optimization problem be denoted as L_1^*, ultimately:

$$L_1 = \begin{cases} \bar{L} & \text{if } L_1^* > \bar{L} \\ L_1^* & \text{if } 0 \leq L_1^* \leq \bar{L} \\ 0 & \text{if } L_1^* 0 \end{cases}$$

The line OAB depicts outcomes when $\mu_1 > \mu_2$ and $\rho < \sigma_2/\sigma_1$. The higher profits of crop 1 may lead to specialization of small farmers if risk is not so high. Larger farms will grow both crops. However when the profit correlation is not sufficiently large, the production of crop 1 will grow absolutely and its land share will decline with size.

The line OAC depicts a situation when $\mu_1 > \mu_2$ and $\rho > \sigma_2/\sigma_1$. In this case correlation is so large that, beyond a certain size, acreage of crop 1 declines with size.

The line ODEF depicts a situation when $\mu_2 > \mu_1$ but $\sigma_1^2 < \sigma_2^2$. When crop 1 is less risky but less profitable on average, it may not be grown by small farmers but may be added to the portfolio of larger ones. It may become the major crop of some very big operators.

In this analysis, risk was the only reason for diversification, and we ignored other constraints facing farmers. There are many situations when other factors, besides risk, cause diversification. They may include labor or equipment scarcity. Growers may grow several crops to spread the harvesting season, thus overcoming labor or capital constraints. Credit limitation may provide another reason. The high value crop may require more credits and that may limit a farmer. Economists have a tendency to attribute "too much" to risk considerations and to ignore those other factors. Realistic analysis has to study in detail local conditions and to incorporate the relevant constraint before conducting allocative investigation.

The framework presented here can be extended to other choices. It is a variation of financial portfolio analysis which is used to investigate distribution of wealth among assets and analysis of financial investments. Similarly, it applies to time allocation analysis including land diversification between on–farm and off–farm activities and migration decisions (time allocation between locations).

5.4.6 Stochastic Dominance

Where the mean-variance and MOTAD approaches are useful in a planning context, the stochastic dominance rule suggested by Anderson (1974) is useful for the comparison of technologies. In general, the stochastic dominance concept allows us to compare the risk associated with each of two probability distributions and to determine which is preferable under an expected utility framework.

Let A and B be two crops. Let $F_A(x)$ be the cumulative distribution of profits under A, and $F_B(x)$ under B. A and B are defined on the same space. A is first-order stochastic dominant to B if:

$$F_A(x) < F_B(x), \qquad \forall x$$

We say that A is stochastic dominant to B of the second degree if, for every x,

$$\int_{-\infty}^{x} F_A(z)dz < \int_{-\infty}^{x} F_B(z)dz$$

This approach can be used for comparisons of both discrete and continuous risky choices. Its usefulness is limited since not all distributions can be ordered through second-degree stochastic dominance.

5.5 Measuring Risk: Mean-Preserving Spread and Stochastic Dominance

5.5.1 Introduction

Economists have made many attempts to define a good measurement for the riskiness of a prospect. These were developed in an attempt to address the question: Given a set of choices, which will a risk–averse individual prefer and how will risk affect her choices? Several general measures were developed within the context of expected utility. Rothschild and Stiglitz (1970) provided a methodology for the ranking of prospects which have the same mean outcome but different levels of risk. In addition, their method provides comparative statics results describing the impact of risk on key parameters. Hadar and Russell (1969) developed a method for ranking prospects with differing mean outcomes. Both of these methods are set within the framework of expected utility, ranking prospects derived from a Von Neumann–Morgenstern concave utility function, implying risk–averse behavior.

5.5.2 Mean-Preserving Spread

Rothschild and Stiglitz (1970) put forth two important concepts in comparing prospects with like means. These are:

1. For any X, Y with $E(X) = E(Y)$, if $EU(X) \geq EU(Y)$ for every U with $U > 0, U" < 0$, then Y is riskier than X. In other words, if every risk–averse individual prefers X to Y, then Y is riskier than X.

2. If $Y \overset{d}{\Leftrightarrow} X + Z$ (i.e., Y is equivalent in distribution to X plus Z) where Z is a random variable with $E(Z|x) = 0$, then Y is riskier than X. In other words, if Y is the sum of X and another random variable Z, when the conditional expectation of Z for every X is zero, Y is riskier than X.

3. If Y can be constructed from X through the use of a *mean–preserving spread*, then Y is riskier than X. A mean–preserving spread is a modification of a distribution that increases the variance of a distribution without changing its mean.

A numerical example of how a mean-preserving spread works may help to shed some light. Given a random variable X, with the following probability distribution:

x	P(X=x)
1	1/8
2	1/8
3	1/2
4	1/8
5	1/8

A new random variable Y is generated from X through use of a mean-preserving spread such that:

x	Change in P(X=x)
1	+1/8
2	-1/8
4	-1/8
5	+1/8

The variable Y then has the following probability distribution:

y	P(Y=y)
1	1/4
3	1/2
5	1/4

According to Rothschild and Stiglitz, iff Y is constructed by a sequence of a mean–preserving spread from X, then $EU(X) > EU(Y)$, for $U' > 0$, $U'' < 0$.

To illustrate the last point consider the above example where we moved from X to Y by a mean-preserving spread. In that case,

$$EU(X) = \left\{\frac{1}{8}U(1) + \frac{1}{8}U(2) + \frac{1}{2}U(3) + \frac{1}{8}U(4) + \frac{1}{8}U(5)\right\}$$

$$EU(Y) = \left\{\frac{1}{4}U(1) + \frac{1}{2}U(3) + \frac{1}{4}U(5)\right\}$$

$$EU(X) \Leftrightarrow EU(Y) = \frac{1}{8}[U(2) \Leftrightarrow U(1) + U(4) \Leftrightarrow U(5)]$$

which is positive due to a decreasing marginal utility under risk-averse behavior, Thus, $U(2) \Leftrightarrow U(1) > U(5) \Leftrightarrow U(4)$, and the alternative X is preferred to Y.

Why is the concept of the mean-preserving spread important? Because it allows for an analysis of the marginal impact of risk. For example, given a production system such that:

$$\Pi = PY \Leftrightarrow WX$$

we can use the mean-preserving spread to manipulate P and create a new variable with the same mean but different variance.

$$P_1 = \gamma P + Z$$

where

$$E(Z) = (1 \Leftrightarrow \gamma)E(P)$$

We can then calculate $\frac{dY}{d\gamma}$ to determine the impact of price risk on output.

5.5.3 Stochastic Dominance

Hadar and Russell (1969) developed the concept of stochastic dominance which allows for the comparison of outcomes with differing means. This method also allows for the ranking of uncertain prospects without assuming a specific utility function.

First-Order Stochastic Dominance (FSD)

Let X denote the value of a variable (wealth, income) which can assume values in the range $(\Leftrightarrow\infty, \infty)$. Let the function $F_i^X(x)$, be defined recursively where

$$F_i^X(x) = \int_{-\infty}^{x} F_{i-1}^X(Z)dZ$$

and where $F_0^X(x)$ is the density function of X.

Under this definition, $F_1^X(x)$ is the cumulative distribution of X. Consider two random prospects, X and Y. There is a first-order stochastic dominance of Y by X, (X FSD Y) if

$$F_1^X(Z) \leq F_1^Y(Z) \quad \text{for } \Leftrightarrow\infty < Z < \infty$$

with at least one point of strong inequality. We can illustrate it graphically

(figure risk1 here)

X is first–order stochastic dominant to Y if the cumulative distribution of X is below that of Y for every Z. That means that Prob$(X \geq Z) >$ Prob$(Y \geq Z)$ for every Z.

It is easy to argue that, if X FSD Y, then every individual with positive marginal utility will prefer X to Y. That is, if Z represents income, then every individual will prefer X over Y since it represents a higher probability of achieving a high income.

Second-Order Stochastic Dominance (SSD)

SSD is a weaker domination concept than FSD which is useful when there is some cross–over between the CDF's of two prospects, but the area under one is greater than the other.

The graph depicts cumulative distributions under SSD

(graph risk2 here)

The cumulative functions in this case intersect twice, with $F_1^Y(Z) < F_1^X(Z)$ in the range between the intersection points. X is SSD to Y if

$$\int_{-\infty}^{Z} F^Y(z) \geq \int_{-\infty}^{Z} F^X(z)$$

SSD relies upon the condition that individuals are strictly risk averse or have diminishing marginal utility (i.e., a concave utility function). Under these assumptions, individuals will prefer X to Y.

Beyond the second order, stochastic dominance is much less useful as a concept. It is unclear what choices a risk–averse individual will make in the presence of third or greater order stochastic dominance. Very restrictive assumptions about behavior are required in order to derive results from higher orders or stochastic dominance.

When stochastic dominance tests are applicable, they are very useful, but they are not applicable in every situation. They present a “partial” ordering ranking only some of the prospects while leaving others out. In particular, stochastic dominance may not capture the trade–offs between risks and returns. An alternative could have a higher mean (non–first degree stochastically dominated) with a higher variance and be second-degree stochastically dominated. The choice between the two alternatives would depend on the degree of risk aversion and the magnitudes of the mean and variance.

5.5.4 Variance as a Measure of Risk

Is variance a good measure of risk? Consider two variables with the following probability distributions:

$$X = \begin{cases} \frac{n+1}{n} & \text{with probability } \frac{n-1}{n} \\ \frac{1}{n} & \text{with probability } \frac{1}{n} \end{cases}$$

$$Y = \begin{cases} \frac{n-1}{n} & \text{with probability } \frac{n-1}{n} \\ \frac{2n-1}{n} & \text{with probability } \frac{1}{n} \end{cases}$$

The means and variances of the two distributions will be as follows:

$$E(X) = U(Y) = 1$$

$$V(X) = V(Y) = \frac{n \Leftrightarrow 1}{n^2}$$

However, even though the means and variances are the same, many risk-averse individuals will prefer Y to X. For this reason, variance is *not* a good measure of risk, although it is often used as a proxy because it is easy to calculate and the data necessary are usually available.

5.5.5 Just and Pope Production Model

Just and Pope (1977) developed general models for analyzing cases of production risk econometrically. Before their work, one had the option of assuming either:

Additive risk: $y = f(x) + \varepsilon, \qquad \text{with } E(\varepsilon) = 0$

Multiplicative risk: $y = f(x)\varepsilon, \qquad \text{with } E(\varepsilon) = 1$

Risk in the production function causes difficulties in the use of linear programming estimation procedures since an investigation into the properties of the random variables is required. Both the additive and the multiplicative models are criticized heavily by Just and Pope. The additive specification does not allow the uncertainty effect to be correlated with the input mix. The multiplicative specification does not allow for inputs that have differing impacts on mean and variance, and for inputs that are yield increasing and risk decreasing such as, for example, fertilizer.

Therefore, they suggest

$$y = f_1(x) + f_2(x)\varepsilon, \qquad \text{with } E(\varepsilon) = 0$$

which offers greater flexibility in describing stochastic technologic processes and related behavior. By including two components in the production function, one relating to output level, and one relating to variability of output, the Just and Pope production model allows for the differential impacts of an input on output and risk. The shortcomings of this model are that it only allows for the consideration of one random variable, and it requires a large number of estimations, including the production function, the risk element, and the behavioral element.

5.5.6 Exponential Utility

Another approach to estimating behavior under risk calls for the use of a utility function that is applicable to any distribution. Yassour and Zilberman (1981) adopted this approach and used an exponential utility function in order to look at farm technology adoption decisions. This utility function can be applied conveniently in conjunction with all distributions which have moment–generating functions. The utility function is written as:

$$U(x) = e^{-rx}$$

and the expected utility function is

$$EU(x) = \Leftrightarrow EU(e^{-rx}) = M(\Leftrightarrow r)$$

where M is a moment–generating function and $E\{e^{tx}\}$ is a well-behaved function. Since the moment-generating function is a function of the parameters, the utility function can be expressed in terms of this function. One shortcoming of this approach is that it implies constant rather than decreasing absolute risk aversion.

Expo–Power Utility

Saha (1993) proposed a new utility function which allows for flexibility in the modeling of risk preference structures. This form allows the data to reveal both the degree and structure (i.e., increasing, constant, or decreasing) of risk aversion. Use of this utility function means that no a priori assumptions about risk preferences are necessary.

The form Saha proposes is called the expo–power function and its formula is:

$$U(w) = \theta \Leftrightarrow \exp\{\Leftrightarrow \beta w^{\alpha}\}$$

where

$$\theta > 1, \quad \alpha \neq 0, \quad \beta \neq 0, \quad \alpha\beta > 0$$

The properties of the expo–power utility function are:

1. It is unique up to an affine transformation.
2. $\Leftrightarrow \frac{U''}{U'} = \frac{1+\alpha\beta w^{\alpha}}{w}$ and $\Leftrightarrow \frac{U''}{U'} w = 1 + \alpha\beta w^{\alpha}$
3. When α is less than one, there is decreasing absolute risk aversion; when α is equal to one, there is constant absolute risk aversion; and when α is greater than one there is increasing absolute risk aversion.
4. When β is less than zero, there is decreasing relative risk aversion; when β is greater than zero there is increasing relative risk aversion;
5. The utility function is quasi-concave for all $w > 0$.

The parameters, α and β, are the key determinants of the risk preference structure. An increase in β leads to an increase in $A(w)$ —the coefficient of absolute risk aversion. The effect of α on $A(w)$ and $R(w)$ —the coefficients of absolute and relative risk aversion— depends on the relative magnitude of w and the parameters.

5.5.7 Summary

Expected utility has theoretical usefulness —such as in Sandmo's results and in the mean variance model— and practical usefulness in applications through either the mean variance model, the MOTAD programming model, or the Just-Pope production function.

Some of the shortcomings of these models have been addressed by the work of Yassour and Zilberman (1981) and Saha (1993). The former allows for the consideration of nonnormal yield distributions in an easy to manipulate format. The latter proposes a flexible utility form with no a priori assumptions about risk preferences specified. Further work extending and testing the results of these models is needed. A major problem is that, up to now, only one variable (either yield or price) is considered random. Many times, both are random. When there are two random variables, only the log-normal distribution yields analytic results.

5.6 Alternatives to Expected Utility: safety rules

Safety rules correspond to expected utility in the same way that classical statistics relate to Bayesian. Safety rules are simple, reasonable, but somewhat arbitrary. They imply lexicographic rules and an objective function that is linear in mean and variance. They fall into three categories as follows:

Roy Minimum Probability Rule

Decision-makers following this rule will choose a production plan subject to minimizing the probability of yield falling below a threshold level D.

$$\min_x \text{Prob}(\Pi(x) < D)$$

where D is a pre–specified disaster level. This approach corresponds to a lexicographic utility model. This rule is useful in modeling subsistence agriculture.

Telser safety-first Rule: A Limit on Disaster Probability

Decision-makers following this rule will maximize expected profit subject to a bound on the probability of profit falling below the threshold level D.

$$\max_x E(\Pi(x))$$

subject to

$$\text{Prob}(\Pi(x) < D) \leq P^*$$

This rule has application in the development of many engineering and health codes and is based on the concept of an onerous event in a manner much like the use of critical levels in Classical Statistics.

Katoka Safety-Fixed Rule

Under this rule decision-makers will maximize the threshold level attained at a given probability level, P^*

$$\max_x D$$

subject to:

$$\text{Prob}(\Pi(x) < D) \geq P^*$$

For normal distributions of outcomes, safety fixed is identical to finding the maximum of the expression

$$\bar{\Pi} \Leftrightarrow b\sigma$$

where $\bar{\Pi}$ and σ are the mean and standard deviation of the outcomes distribution, subject to a probability level constraint. Under normality, we choose a line of actions to maximize D subject to the constraint:

$$\text{Prob}(\frac{\Pi(x) \Leftrightarrow \bar{\Pi}}{\sigma} < \frac{D \Leftrightarrow \bar{\Pi}}{\sigma}) \geq P^*$$

If the constraint is binding, this amount to choose D such that

$$\frac{D \Leftrightarrow \bar{\Pi}}{\sigma} = Z_{P^*}$$

where Z_{P^*} is such that $Prob(x < Z_{P^*}) = P^*$ under the standard normal distribution.

Thus, $D = \bar{\Pi} + Z_{P^*}\sigma$, and maximizing D, in the case of the normal distribution, amounts to maximize the sum of the mean and a multiple of the standard deviation.

The advantages of safety rules are:

1. They can be used with programming techniques.
2. They follow the logic of conventional statistics while expected utility is Bayesian.
3. They are frequently used by engineers and regulators in constructing nuclear power plants and devising earthquake regulations.
4. They provide a satisfying model of behavior in many circumstances.

These models are useful both in positive and normative analyses. For applications in development, see Roumasset and de Janvry and Moscardi (1977).

5.7 Safety Rules

5.7.1 Introduction

The main appeal of the expected–utility approach to modeling decisions under uncertainty is that it is derived rigorously from a well defined and reasonable set of assumptions about preferences. One disadvantage of this approach is the assumption that individuals know the probabilities associated with each possible outcome of a prospect, and that they have based theft decision upon this knowledge In many cases, the degree of information and computation required may make his assumption unrealistic. An alternative approach for modeling choices under certainty is embodied by the various “safety rules”.

“Safety rules” reflect a “behavioristic” approach to modeling behavior. The models are based upon simple decision criteria economists believe people use in making day–to–day decisions. One common element of these rules is that they reflect "satisfying" behavior; where people make choices to meet some objective. Thus, under this approach, people do not “maximize”, rather, they aim to meet a target. A major proponent of this approach was Herbert Simon, a Nobel Prize winner in Economics. Simon introduced the notion of “bounded rationality”, where people are constrained by high computation information and competition costs, and therefore, they develop simple decision rules. Simon saw these behavioristic rules as outcomes of optimization subject to all computation and data costs and constraints, and his views pose a challenge to economists to identify the constrained optimization problems that have resulted in the behavioristic rules whose existence is supported statistically.

When economists identify persistent behavioral rules, they can be used for prediction and analysis of appropriate policy. They can be particularly relevant in setting government regulations.

The safety rules that we present are intuitively appealing and are likely to represent the behavior of at least some people. They correspond in their structure to decision rules of “classical” statistics. In particular, they resemble the use of statistical significance in classical “hypothesis testing”.

5.7.2 Roy Minimum Probability Rule

Let π, denoting profit, be a random variable whose distribution depends on a decision variable, X. For example, $\pi = PX \Leftrightarrow C(X)$ where P is a random variable. One safety rule is the Safety First approach introduced by Roy (1952). Under this approach, the objective function is

$$\max_X Prob\{\Pi(X) \geq D\} \text{ or } \min_X Prob\{\Pi(X) < D\}$$

where D is a disaster level. Under this approach, agents' choices are dominated by the desire to minimize the probability of bankruptcy. If $P \sim N(\bar{P}, \sigma^2)$, then Roy's safety rule suggests setting the X that minimizes the probability that a standard–normal random variable is smaller than $\frac{D-PX-C(X)}{\sigma X}$.

If the cumulative distribution of the standard normal random variable is denoted by $\Phi(Z)$, then the Roy rule is

$$X^* = \frac{D \Leftrightarrow \bar{P}X \Leftrightarrow C(X)}{\sigma X}$$

(figure risk3 here)

5.7.3 Telser Safety-First Rule

An alternative approach is provided by Telser (1955). In his case, the objective is maximized expected profit when the probability to be under a disaster level is contained below a significance level, α

This model is

$$\max_X E\pi(X) \quad \text{s. t.} Prob\{\pi(x) \leq D\} < \alpha$$

For our previous example with a normal distribution, the Telser rule is

$$\max_X \bar{P}X \Leftrightarrow C(X) \quad \text{s. t.} Prob\{Z \leq \frac{D \Leftrightarrow \bar{P}X \Leftrightarrow C(X)}{\sigma X}\} < \alpha$$

where $Z \sim N(0,1)$ is a standardized normally distributed random variable

5.7.4 Safety Fixed

Kataoka (1963) developed the safety-first rule which maximizes a minimum profit level which can be obtained with a probability of at least the critical value α. Mathematically, the safety-fixed rule can be presented as

$$\max_X D \quad \text{s.t.} Prob\{\pi(X) < D\} < \alpha$$

This safety rule and the safety-first rule are inversely related. The statistical significance level α is the parameter of the safety fixed rule, and its objective is to find the profit distribution which has the highest level of profit when the cumulative distribution is at α. The disaster level D is the parameter of the safety–first rule, and its objective is to select the profit distribution with the lowest level of cumulative distribution when profit is equal to D. Figure (5.7.4) demonstrates the two rules graphically.

(figure risk3 here)

Suppose we have to choose among three activities. The cumulative distributions of profits of the three activities are denoted by the F_i functions in Figure (5.7.4). Activity 1 is selected under the safety–fixed rule when α is α_1. It is selected under safety–first when D is D_1. When the safety–fixed rule is not as restrictive and α is α_2, activity 2 is selected and this activity is selected under safety first when D is D_2.

5.7.5 Safety Rules and Expected Utility

Safety rules can be expressed as special cases of the expected utility framework. The safety-first rule can be expressed as the outcome of the expected utility framework when the utility function is lexicographic. The objective of the safety–first rule is to minimize the probability of profits below the disaster level D. That is equivalent to expected utility maximization when the utility function is

$$U(\pi) = \begin{cases} 1 & \text{when } \pi \geq D \\ 0 & \text{when } \pi < D \end{cases}$$

With this utility function, the probability that profit is greater than D is maximized. The main criticism of the safety-first approach is that lexicographic utility functions are unreasonable, and that individuals prefer making more money than less once their profits are above the disaster level. There are likely to be situations where the safety rules are good approximators of reality, and their use is justified. One major flaw of the standard expected utility models is that they assume that utility is a function of income only. There are many situations where utility depends on other variables, such as social status, (e.g., landlord vs. landless peasant; hired hand vs. self–employed, etc.). If social status is lost when profits are below a threshold level, and such a status loss entails a drastic reduction in welfare, when activities result in profits level in the neighborhood of the threshold level, the use of safety first provides a good approximation of expected utility outcomes.

For some profit distributions, safety rules may result in many interesting decision rules. For example, when profits have a normal distribution with mean μ and variance σ^2, the safety-fixed rule results in the maximization of a linear combination of the mean and standard deviation of profits. To see this note, that when $\pi \sim N(\mu, \sigma^2)$

$$\max D \quad \text{s. t.} Prob\{\pi \leq D\} = \alpha$$

yields under normality

$$\max D \quad \text{s. t.} Prob\left\{ Z \leq \frac{D \Leftrightarrow \mu}{S} \right\} = \alpha \Rightarrow \max \mu + Z_\alpha S$$

where Z_α is the value of a standardized normal random variable that is not exceeded with probability of α. Thus, Z_α is the coefficient of the standard deviation in a linear combination of mean and standard deviation when profits is maximized under safety fixed rules.

For the case with

$$\mu = \bar{P}X \Leftrightarrow c(X)$$

and

$$S = \sigma X$$

the safety–fixed objective function becomes

$$\max PX \Leftrightarrow C(X) + Z\alpha\sigma X$$

when $\alpha < .5$, $2\alpha < 1$ and we have some form of risk aversion.

5.8 Prospect Theory

The attractive feature of expected utility is that it is derived from a set of axioms about human behavior. While this set is reasonable, rational, and very desirable as a base to normative analysis, it is sometimes criticized as unrealistic. The following experimental examples show some instances of when individuals violate expected utility.

Kahnman and Tversky Experiments

Kahnman and Tversky conducted experiments to test the validity of the expected utility model. Individuals were asked to select one of two options from the following choice pairs:

	Choice 1			Choice 2		
(1)	X	$P(X)$	chosen	X	$P(X)$	chosen
A	4,000	.80	20%	4,000	.20	65%
B	3,000	1	80%	3,000	.25	35%
	Choice 3			Choice 4		
A	-4,000	.80	92%	-4,000	.20	42%
B	-3,000	1	8%	-3,000	.25	58%

IMPLICATIONS from (1): $U(3,000) > .8U(4,000)$

IMPLICATIONS from (2): $.25U(3,000) < .2U(4,000)$,
or $U(3,000) < .8U(4,000)$

IMPLICATIONS from (3): $.8U(\Leftrightarrow 4,000) > U(\Leftrightarrow 3,000)$

IMPLICATIONS from (4): $.2U(\Leftrightarrow 4,000) < .25U(\Leftrightarrow 3,000)$,
or $.8U(\Leftrightarrow 4,000) < U(\Leftrightarrow 3,000)$

Therefore, the choices for the pairs in questions (1) and (2), and those for the pairs in questions (3) and (4) are inconsistent with the predictions of the expected utility model.

Conclusion –The validity or Expected Utility is cast into doubt

Two major explanations for the apparent failure of the expected utility model to correctly predict human behavior have been advanced. These are:

1. People under–weigh outcomes that are probable in comparison to certain outcomes, even when they both have the same expected utility, which is known as the *certainty effect*.

2. People are risk averse when they gain risk and risk lovers when they lose; known as the *reflection effect*.

5.9 The Validity of the Expected Utility (EU) Model

The EU model has been widely embraced by economists for modeling decisions under risk because of its ease of use, normative appeal, and (it is argued) for its reasonable accuracy in predicting behavior under risk for many economic activities.

However, the EU model has repeatedly been sbown to lack descriptive and predictive validity in experimental settings. These empirical violations of EU have given rise to the formulation of a large number of alternative Generalized-EU models, even though little is known about the underlying reasons for the occurrence of EU violations.

Applied economists face a dilemma when choosing between models of decision making under risk. They must chose between: (1) the EU model that has normative appeal but has been shown to be systematically violated by behavior or (2) one of a number of generalized models that lack normative appeal and allow for some behavioral violations of EU. These notes offer a brief discussion of the EU model's implications for behavior, the experimental violations of EU, the approach of the generalized-EU models, and an explanation for the occurrence of choices violating EU which offers direction to applied economists for model selection in risky choice environments.

5.9.1 Background: The EU Model's Critical Implications for Behavior

There are three main axioms in the EU framework. They are defined over a binary relation where:

$\succeq$ denotes weak preference,

$\succ$ denotes strong preference, and

$\sim$ denotes indifference.

for preferences over probability distributions $p, q \in P$ that are defined over a common (discrete or continuous) outcome vector $\mathbf{x}$. The three axioms that are necessary and sufficient for the EU representation $u(\cdot)$ over preferences (cf., Jensen, Fishburn) are:

Axiom O (Order):
The preference ordering $\succeq$ is a weak ordering.
This axiom implies that the preference ordering is complete (all distributions $p, q \in P$ are comparable via the ordering) and transitive (if $p \succeq q$, and $q \succeq r$, then $p \succeq r$).

Axiom I (Independence):
For all $p, q, r \in P$, and for all $\alpha \in (0, 1)$, if $p \succeq q$, then $\alpha p + (1 \Leftrightarrow \alpha) r \succeq \alpha q + (1 \Leftrightarrow \alpha) r$
This axiom holds that preferences over probability distributions should only depend on the portions of the distributions that differ (p and q), not on their common elements (r). This independence of preference with respeet to r holds regardless of r and of the level of α that defines the linear combination.

Axiom C (Continuity):
For all $p, q, r \in P$ with $p \succeq q$ and $q \succeq r$, there exist $\alpha, \beta \in (0, 1)$ such that:

$$\alpha p + (1 \Leftrightarrow \alpha) r \succeq q \text{ and } q \succeq \beta p + (1 \Leftrightarrow \beta) r$$

This axiom gives a degree of continuity to the preferences.

Axioms **O**, **I**, and **C** can be shown to be necessary and sufficient (Jensen; Fishburn) for the existence of a function $u(\cdot)$ on the outcomes $x \in X$ that represents preferences through $\succeq$. In the discrete case where $p = \{p_1, p_2, \ldots, p_n\}$, it gives the probabilities of occurrence for $x = \{x_1, x_2, \ldots, x_n\}$:

$$p \succeq q \Leftrightarrow \sum_{i=1}^{n} u(x_i)p_i \geq \sum_{i=1}^{n} u(x_i)q_i$$

Most of the violations of EU hinge on an implication stemming primarily from the Independence Axiom.

A useful diagram for viewing the implications for models of behavior under risk was developed by Marschak and reintroduced by Machina (Figure 1). This triangle in two dimensions has boundaries from 0 to 1 (a simplex) consistent with rules of probability.

Probability distributions defining gambles over three (low, medium, high) discrete outcomes are represented by points inside or on the boundaries of this triangle. The distribution's probability for the occurrence of the lowest outcome (x_L) is given on the horizontal axis, the probability of occurrence for the highest outcome (x_H) is given on the vertical axis, and the probability of the medium outcome (x_M) is given implicitly by 1 less the sum of the probabilities for the high and the low outcomes.

To further explain, points on the hypotenuse represent gambles with no choice of occurrence for the middle outcome (the sum of the probabilitics for the low and the high outcomes is one), while the point on the vertix of the triangle opposite to the hypotenuse represents a gamble giving the middle outcome with certainty.

Preferences over gambles can be illustrated by indifference curves within the triangle; the EU model holds that these indifference curves must be parallel as in Figure 1. These indifference curves can be compared with the equi-expected value lines in the figure, where the risky alternatives have the same expected values. The indifference curves shown in Figure 1 indicate that the individual is risk averse, as increased risk (movements along an indifference curve to the "Northeast") require a higher expected value for indifference to hold. Individuals prefer movements toward the "Northwest" of the triangle, as the probability of the highest outcome increases while that of the lowest outcome decreases.

Violations of EU

There have been many advances in the economic analysis of decisions under risk using the EU model. (e.g., Friedman and Savage; Sandmo; Newbery and Stiglitz). These papers all take the validity of the EU model as given. A serious challenge to the use of EU was made by the Nobel prize (in economics) winner Maurice Allais soon after its introduction. His work and that of others following him elicited choices between hypothetical risky alternatives to show that EU lacked complete predictive, and hence descriptive, validity.

Some well-known risky choice examples are given in a paper by Kahneman and Tversky (1979) that synthesizes work by Allais and by others who have shown experimental violations of EU. This paper also presents a model of choice which strives for only descriptive (not normative) validity. Kahneman and Tversky's paper remains a standard in this subject of modeling choice under risk; in particular, their experimental results have had a great deal of influence on the literature.

The first of Kahneman and Tversky's examples showing EU violations discussed here (shown in Figure 2) asks individuals to select between two pairs of risky alternatives, where one of the alternatives in the pair (A and C, respectively) is less risky than the other alternative that has a higher expected value (B and D, respectively)

Kahneman and Tversky's Choice 1: Select between A and B

Gamble A	Gamble B
\$3000 with probability 1.0	\$4000 with probability .8, \$0 with probability .2

Kahneman and Tversky's Choice 2: Select between C and D

Gamble C	Gamble D
\$3000 with probability .25, \$0 with probability .75	\$4000 with probability .2 \$0 with probability .8

The EU model requires that the choice between A and B must be compatible with the choice between C and D; i.e., if the more risky alternative B is selected in the first choice, the more risky alternative D must be selected in the second choice and viceversa. One of these choice patterns is required due to the Independence Axiom, since the probability vectors $\{(.75, .25, 0), (.8, 0, .2)\}$ over the outcome vector $x = (\$0, \$3000, \$4000)$ defining alternatives C and D, respectively, can be viewed as a linear combination of the probability distributions $\{p = (0, 1.0, 0), q = (.2 .0 .8)\}$ that define A and B, respectively. In the definitions below that show the relationship between the pairs through the Independence Axiom, a distribution carrying a certain outcome of \$0 is defined by (\$0)

$$.25 \cdot p + .75 \cdot (\$0) = (.75, .25, 0) \quad \text{The probability vector defining C.}$$

$$.25 \cdot q + .75 \cdot (\$0) = (.80, 0, .20) \quad \text{The probabifly vector derming D.}$$

In their experiment using hypothetical payoff outcomes, many (65Kahneman and Tversky's subjects selected A over B in the first pair but sclected D over C in the second pair, a choice pattern that violates EU. Figure 2a illustrates this pattern of choice.

The gamble pairs (A,B) and (C,D) are connected in Figure 2a by loci of points; these parallel lines are important for the analysis of choice with respect to the EU model. The "Indifference Curves #1 and #2" in the figure correspond with preference representations $u_1(x)$, $u_2(x)$ that would be consistent with the choices made for each pair. Of particular note as a result of the construction of the pairs (AB) and (CD) is that these indifference curves cannot be parallel as called for under EU.

Another well-known EU violation was found by Kahneman and Tversky where they asked respondents to select between hypothetical risky alternatives that have equal expected values:

Kahneman and Tversky's Choice 3: Select between E and F.

Gamble E	Gamble F
\$3000 with probability .9,	\$6000 with probability .45,
\$ 0 with probability .1	\$ 0 with probability .55

Choice 4: Select between 0 and H.

Gamble G	Gamble H
\$3000 with probability .002,	\$6000 with probability .001,
\$ 0 with probability .998	\$ 0 with probability .999

The probabilities defining G and H can also be written as a linear combination of the probabilities that define E and F, with (\$0) defining a distribution giving a certain outcome of \$0. The EU model requires consistency of choice: if E is selected in the first pair, G must be selected in the second. On the other hand, if F is selected in the first choice, H must be selected in the second.

Respondents also violated the EU model in their choices over these two pairs, with most respondents selecting E over F and H over G. The gambles and the hypothetical "Indifference Curves #3 and #4" are shown in Figure 2b. The choice of the riskier H over G also violates second-degree stochastic doniinance as H and G have equal expected value.

The Generalized–EU models

A number of models have been set forth as alternatives to EU in light of the behavioral violations of EU such as those developed by Kahneman and Tversky described above. These models weaken the Independence Axiom of EU in order to allow for observed behavioral violations. The upshot of these models is that preferences are represented through both a function $u(x)$ over the outcomes, but also through a nonlinear function $g(s)$ for $s \in \{p, q\}$ over the probability distributions, giving the representation:

$$p \succeq q \Leftrightarrow \sum_{i=1}^{n} u(x_i) g_i(p) \geq \sum_{i=1}^{n} u(x_i) g_i(q)$$

The function $u(x)$ has similar structure as in the EU model; the interesting part of these models is the function $g(r)$ for $r \in P$. Quiggin proposed a model in 1982 (*Journal of Economic Behavior and Organization*) where $g(\cdot)$ overweighted extremely small probabilities when defincd over either very low ($0 in the examples) or relatively high ($4000, $6000 in the examples) outcomes. Recent models by Tversky and Kahneman; by Wakker and by Luce and Fishburn incorporate Quiggin's structure of this function $g(\cdot)$.

Another early, well known, and simple to illustrate form of the Generalized-EU models was developed by Machina (Econometrica, 1982). In this model, preferences locally correspond with EU in the same manner as Taylor's series approximations for nonstochastic functional forms. Machina further specifies the behavior of the preferences in order to allow for the empirical violations of EU through a curvature change in the triangle known as "fanning out". Figure 3 shows fanning out of the nonlinear but smooth preference curves, where the indifference curves become steeper with increases in the expected values of the gamble (movements toward the northwest). This fanning out notion has been used for other Generalized-EU models, but is neither strongly motivated nor predictively accurate.

The Similarity Model: Alternative Explanation for the Paradoxes

An appealing explanation of the patterns of choices showing inconsistencies with EU proposes that individuals evaluate risky alternatives differently, dependent on the similarity of the alternatives. This similarity has both objective and subjective connotations. In risky choice, selection between the more similar alternatives would likely be both (1) more difficult or (mentally) costly and (2) less beneficial or important since the alternatives differ little in an objective sense. As a result, comparisons between two sets of choice pairs that differ considerably in their degree of similarity (such as those used to show violations of EU) may give misleading implications about preferences in the nature of model misspecification, since both preferences and perceplions are reflected in choices.

To illustrate the application of this similarity idea, consider the pairs of risky alternatives in the Certainty Effect and the Common Ratio Effect examples discussed above and illustrated in Figures 2a and 2b, respectively. In both of these examples, one of the choice pairs [(AB) in the Cenainty Effect and (EF) in the Common Ratio Effect] is quite "dissimilar" as defined by the distance (Euclidian norm) metric over the probability space;

$$\text{metric}(p,q) = \left[\sum_{i=1}^{n}(p_i \Leftrightarrow q_i)^2\right]^{\frac{1}{2}}$$

In addition to distance differences, the pair (AB) is qualitatively different, since A gives $3000 with certainty (no risk).

Choices between these dissindiar pairs are compared with choices between similar pairs [(CD) in the Certainty Effect and (GH) in the Common Ratio Effect]. Kahneman and Tversky's results can be explained if individuals are more likely to select the riskier (D and H in the Cenainty Effect and Common Ratio Effect, respectively) alternative when the alternatives are similar.

Work by Leland, and by Rubinstein, has explored some of the implications of various models of similarity on risky choice, although the models suggested by these authors are quite limited in their applications. Buschena and Zilberman have developed and tested more general models for the similarity of risky choice and have found considerable effects of similarity on both the pattern of choice and on the occurrence of EU violabons. The tests show two primary results. First, as the risky choice pair becomes more similar (the choice is less critical and the evaluation is more costly), the riskier alternative is much more likely to be selected. Second, violations of EU are much more likely to occur when the differences in the dissimilarity between the two sets of risky choice pairs is large, i.e., when there is a good deal of dichotomy between the dissimilarity of the pairs.

The results of the similarity tests in Buschena and Zilberman show some results that should be of significant interest to general and applied economists, namely that:

1. There is an operational and intuitively appealing explanation for the occurrence of choice patterns that violate EU.

2. There are a significant number of decisions over risky alternatives for which the EU model is descriptively accurate; i.e., EU works well for dissimilar pairs.

3. Statistical analysis shows significant violations of the models set forth as alternatives to EU (Generalized–EU and others); moreover, these violations were in the direction predicted by the Similarity Model.

There remain a number of unanswered questions regarding the effects of similarity on risky choice. Of particular note are those concerning the effects of using real, rather than hypothetical, payoffs on the influence of similarity on choice. Further questions of interest that are quite unexplored are the occurrence of EU violations themslves, and also the effects of similarity on the occurrence of violations, for nonexperimental choices (e.g., agricultural production decisions and risky resource protection issues). There is some evidence (Bar-Shira, *American Journal of Agricultural Economics*, 1992) that the EU model works well for modeling crop portfolio choices.

Research modeling and testing the robustness of the findings of experiments to risky decisions made in "everyday life" is a very fertile one. This line of research, however, will likely be quite difficult to carry out given data availability and calls for a good deal of creativity in experimental design. Some of the recent findings discussed here indicate promise for models of behavior incorporating factors reflective of more comprehensive models of decision making.