

## CHAPTER 3

### APPENDIX

## THE UTILITY FUNCTION APPROACH TO THE CONSUMER BUDGETING PROBLEM

### *The Utility-Function Approach to Consumer Choice*

Finding the highest attainable indifference curve on a budget constraint is just one way that economists have analyzed the consumer choice problem. For many applications, a second approach has also proved useful. In this approach we represent the consumer's preferences not with an indifference map but with a *utility function* (a formula that yields a number representing the satisfaction provided by a bundle of goods.)

A utility function is simply a formula that, for each possible bundle of goods, yields a number that represents the amount of satisfaction provided by that bundle. Suppose, for example, that Tom consumes only food and shelter and that his utility function is given by  $U(F, S) = FS$ , where  $F$  denotes the number of pounds of food,  $S$  the number of square yards of shelter he consumes per week, and  $U$  his satisfaction, measured in “utils” per week.<sup>1</sup> If  $F = 4$  lb/wk. and  $S = 3$  sq yd/wk, Tom will receive 12 utils/wk of utility, just as he would if he consumed 3 lb/wk of food and 4 sq yd/wk of shelter. By contrast, if he consumed 8 lb/wk of food and 6 sq yd/wk of shelter, he would receive 48 utils/wk.

The utility function is analogous to an indifference map in that both provide a complete description of the consumer's preference ordering. In the indifference curve framework, we can rank any two bundles by seeing which one lies on a higher indifference curve. In the utility-function framework, we can compare any two bundles by seeing which one yields a greater number of utils. Indeed, as the following example illustrates, it is straightforward to use the utility function to construct an indifference map.

#### **Example A.3-1**

*If Tom's utility function is given by  $U(F, S) = FS$ , graph the indifference curves that correspond to 1, 2, 3, and 4 utils, respectively.*

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In the language of utility functions, an indifference curve is all combinations of  $F$  and  $S$  that yield the same level of utility—the same number of utils. Suppose we look at the indifference curve that corresponds to 1 unit of utility—that is, the combinations of bundles for which  $FS = 1$ . Solving this equation for  $S$ , we have

$$S = \frac{1}{F}, \quad (\text{A.3.1})$$

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<sup>1</sup>The term “utils” represents an arbitrary unit. As we will see, what is important for consumer choice is not the actual number of utils various bundles provide, but the rankings of the bundles based on their associated utilities.

which is the indifference curve labeled  $U = 1$  in Figure A.3-1. The indifference curve that corresponds to 2 units of utility is generated by solving  $FS = 2$  to get  $S = 2/F$ , and it is shown by the curve labeled  $U = 2$  in Figure A.3-1. In similar fashion, we generate the indifference curves to  $U = 3$  and  $U = 4$ , which are correspondingly labeled in the diagram. More generally, we get the indifference curve corresponding to a utility level of  $U_0$  by solving  $FS = U_0$  to get  $S = U_0/F$ .

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[Figure A.3-1]

In the indifference-curve framework, the best attainable bundle is the bundle on the budget constraint that lies on the highest indifference curve. Analogously, the best attainable bundle in the utility-function framework is the bundle on the budget constraint that provides the highest level of utility. In the indifference-curve framework, the best attainable bundle occurs at a point of tangency between an indifference curve and the budget constraint. At the optimal bundle, the slope of the indifference curve, or MRS, equals the slope of the budget constraint. Suppose food and shelter are again our two goods, and  $P_F$  and  $P_S$  are their respective prices. If  $\Delta S/\Delta F$  denotes the slope of the highest attainable indifference curve at the optimal bundle, the tangency condition says that  $\Delta S/\Delta F = P_F/P_S$ . What is the analogous condition in the utility-function framework?

To answer this question, we must introduce the concept of *marginal utility* (the marginal utility of a good is the rate at which total utility changes with consumption of the good) which is the rate at which total utility changes as the quantities of food and shelter change. More specifically, let  $MU_F$  denote the number of additional utils we get for each additional unit of food and  $MU_S$  denote the number of additional utils we get for each additional unit of shelter. In Figure 3-2, note that bundle  $K$  has  $\Delta F$  fewer units of food and  $\Delta S$  more units of shelter than bundle  $L$ . Thus, if we move from bundle  $K$  to bundle  $L$ , we gain  $MU_F\Delta F$  utils from having more food, but we lose  $MU_S\Delta S$  utils from having less shelter.

[Figure A.3-2]

Because  $K$  and  $L$  both lie on the same indifference curve, we know that both bundles provide the same level of utility. Thus the utility we lose from having less shelter must be exactly offset by the utility we gain from having more food. This tells us that

$$MU_F\Delta F = MU_S\Delta S. \tag{A.3.2}$$

Cross-multiplying terms in Equation A.3.2 gives

$$\frac{MU_F}{MU_S} = \frac{\Delta S}{\Delta F}. \tag{A.3.3}$$

Suppose that the optimal bundle lies between  $K$  and  $L$ , which are very close together, so that  $\Delta F$  and  $\Delta S$  are both very small. As  $K$  and  $L$  move closer to the optimal bundle, the

ratio  $\Delta S/\Delta F$  becomes equal to the slope of the indifference curve at that bundle, which Equation A.3.3 tells us is equal to the ratio of the marginal utilities of the two goods. And since the slope of the indifference curve at the optimal bundle is the same as that of the budget constraint, the following condition must also hold for the optimal bundle:

$$\frac{MU_F}{MU_S} = \frac{P_F}{P_S}. \quad (\text{A.3.4})$$

Equation A.3.4 is the condition in the utility-function framework that is analogous to the  $MRS = P_F/P_S$  condition in the indifference-curve framework.

If we cross-multiply terms in Equation A.3.4, we get an equivalent condition that has a very straightforward intuitive interpretation:

$$\frac{MU_F}{P_F} = \frac{MU_S}{P_S}. \quad (\text{A.3.5})$$

In words, Equation A.3.5 tells us that the ratio of marginal utility to price must be the same for all goods at the optimal bundle. The following examples illustrate why this condition must be satisfied if the consumer has allocated his budget optimally.

### Example A.3-2

*Suppose that the marginal utility of the last dollar John spends on food is greater than the marginal utility of the last dollar he spends on shelter. For example, suppose the prices of food and shelter are \$1/lb and \$2/sq yd, respectively, and that the corresponding marginal utilities are 6 and 4. Show that John cannot possibly be maximizing his utility.*

If John bought 1 sq yd/wk less shelter, he would save \$2/wk and would lose 4 utils. But this would enable him to buy 2 lb/wk more food, which would add 12 utils, for a net gain of 8 utils.

Abstracting from the special case of corner solutions, a necessary condition for optimal budget allocation is that the last dollar spent on each commodity yield the same increment in utility.

### Example A.3-3

*Mary has a weekly allowance of \$10, all of which she spends on newspapers (N.) and magazines (M), whose respective prices are \$1 and \$2. Her utility from these purchases is given by  $U(N) + V(M)$ . If the values of  $U(N)$  and  $V(M)$  are as shown in the table, is Mary a utility maximizer if she buys 4 magazines and 2 newspapers each week? Of not, how should she reallocate her allowance?*

<b>N</b>	<b>U(N)</b>	<b>M</b>	<b>V(M)</b>
0	0	0	0
1	12	1	20
2	20	2	32
3	26	3	40
4	30	4	44
5	32	5	46

For Mary to be a utility maximizer, extra utility per dollar must be the same for both the last newspaper and the last magazine she purchased. But since the second newspaper provided 8 additional utils per dollar spent, which is four times the 2 utils per dollar she got from the fourth magazine (4 extra utils at a cost of \$2), Mary is not a utility maximizer.

To see clearly how she should reallocate her purchases, let us rewrite the table to include the relevant information on marginal utilities.

<i>N</i>	<i>U(N)</i>	<i>MU(N)</i>	<i>MU(N)/P<sub>N</sub></i>	<i>M</i>	<i>U(M)</i>	<i>MU(M)</i>	<i>MU(M)/P<sub>M</sub></i>
0	0			0	0		
		12	12			20	10
1	12	8	8	1	20	12	6
2	20	6	6	2	32	8	4
3	26	4	4	3	40	4	2
4	30	2	2	4	44	2	1
5	32			5	46		

From this table, we see that there are several bundles for which  $MU(N)/P_N = MU(M)/P_M$ —namely, 3 newspapers and 2 magazines; or 4 newspapers and 3 magazines; or 5 newspapers and 4 magazines. The last of these bundles yields the highest total utility but costs \$13, and is hence beyond Mary's budget constraint. The first, which costs only \$7, is affordable, but so is the second, which costs exactly \$10 and yields higher total utility than the first. With 4 newspapers and 3 magazines, Mary gets 4 utils per dollar from her last purchase in each category. Her total utility is 70 utils, which is 6 more than she got from the original bundle.

In Example A.3-3, note that if all Mary's utility values were doubled, or cut by half, she would still do best to buy 4 newspapers and 3 magazines each week. This illustrates the claim, that consumer choice depends not on the absolute number of utils associated with

different bundles, but instead on the ordinal ranking of the utility levels associated with different bundles. If we double all the utils associated with various bundles, or cut them by half, the ordinal ranking of the bundles will be preserved, and thus the optimal bundle will remain the same. This will also be true if we take the logarithm of the utility function, the square root of it, or add 5 to it, or transform it in any other way that preserves the ordinal ranking of different bundles.

### ***Cardinal versus Ordinal Utility***

In our discussion about how to represent consumer preferences, we assumed that people are able to rank each possible bundle in order of preference. This is called the *ordinal utility* approach to the consumer budgeting problem. It does not require that people be able to make quantitative statements about how much they like various bundles. Thus it assumes that a consumer will always be able to say whether he prefers *A* to *B*, but that he may not be able to make such statements as “*A* is 6.43 times as good as *B*.”

In the nineteenth century, economists commonly assumed that people could make such statements. Today we call theirs the *cardinal utility* approach to the consumer choice problem. In the two-good case, it assumes that the satisfaction provided by any bundle can be assigned a numerical, or cardinal, value by a utility function of the form

$$U = U(X, Y). \tag{A.3.6}$$

In three dimensions, the graph of such a utility function will look something like the one shown in Figure A.3-3. It resembles a mountain, but because of the more-is-better assumption, it is a mountain without a summit. The value on the *U* axis measures the height of the mountain, which continues to increase the more we have of *X* or *Y*.

Suppose in Figure A.3-3 we were to fix utility at some constant amount, say,  $U_0$ . That is, suppose we cut the utility mountain with a plane parallel to the *XY* plane,  $U_0$  units above it. The line labeled *JK* in Figure A.3-3 represents the intersection of that plane and the surface of the utility mountain. All the bundles of goods that lie on *JK* provide a utility level of  $U_0$ . If we then project the line *JK* downward onto the *XY* plane, we have what amounts to the  $U_0$  indifference curve, shown in Figure A.3-.

[Figure A.3-3]

Suppose we then intersect the utility mountain with another plane, this time  $U_1$  units above the *XY* plane. In Figure A.3-3, this second plane intersects the utility mountain along the line labeled *LN*. It represents the set of all bundles that confer the utility level  $U_1$ . Projecting *LN* down onto the *XY* plane, we thus get the indifference curve labeled  $U_1$  in Figure A.3-3. In like fashion, we can generate an entire indifference map corresponding to the cardinal utility function  $U(X, Y)$ .

[Figure A.3-4]

Thus we see that it is possible to start with any cardinal utility function and end up with a unique indifference map. *But it is not possible to go in the other direction!* That is,

it is not possible to start with an indifference map and work backward to a unique cardinal utility function. The reason is that there will always be infinitely many such utility functions that give rise to precisely the same indifference map.

To see why, just imagine that we took the utility function in Equation A.3.4 and doubled it, so that utility is now given by  $V = 2U(X, Y)$ . When we graph  $V$  as a function of  $X$  and  $Y$ , the shape of the resulting utility mountain will be much the same as before. The difference will be that the altitude at any  $X, Y$  point will be twice what it was before. If we pass a plane  $2 U_0$  units above the  $XY$  plane, it would intersect the new utility mountain in precisely the same manner as the plane  $U_0$  units high did originally. If we then project the resulting intersection down onto the  $XY$  plane, it will coincide perfectly with the original  $U_0$  indifference curve.

All we do when we multiply (divide, add to, or subtract from) a cardinal utility function is to relabel the indifference curves to which it gives rise. Indeed, we can make an even more general statement: If  $U(X, Y)$  is any cardinal utility function and if  $V$  is any increasing function, then  $U = U(X, Y)$  and  $V = V[U(X, Y)]$  will give rise to precisely the same indifference maps. The special property of an increasing function is that it preserves the rank ordering of the values of the original function. That is, if  $U(X_1, Y_1) > U(X_2, Y_2)$ , the fact that  $V$  is an increasing function assures that  $V[U(X_1, Y_1)]$  will be greater than  $V[U(X_2, Y_2)]$ . And as long as that requirement is met, the two functions will give rise to exactly the same indifference curves.

The concept of the indifference map was first discussed by Francis Edgeworth, who derived it from a cardinal utility function in the manner described above. It took the combined insights of Vilfredo Pareto, Irving Fisher, and John Hicks to establish that Edgeworth's apparatus was not uniquely dependent on a supporting cardinal utility function. As we have seen, the only aspect of a consumer's preferences that matters in the standard budget allocation problem is the shape and location of his indifference curves. Consumer choice turns out to be completely independent of the labels we assign to these indifference curves, provided only that higher curves correspond to higher levels of utility.

Modern economists prefer the ordinal approach because it rests on much weaker assumptions than the cardinal approach. That is, it is much easier to imagine that people can rank different bundles than to suppose that they can make precise quantitative statements about how much satisfaction each provides.

### ***Generating Indifference Curves Algebraically***

Even if we assume that consumers have only ordinal preference rankings, it will often be convenient to represent those preferences with a cardinal utility index. The advantage is that this procedure provides a compact algebraic way of summarizing all the information that is implicit in the graphical representation of preferences.

[Figure A.3-5]

Suppose, for example, that Tom's utility function is given by  $U(X, Y) = XY$ , and we want to use this information to graph Tom's indifference map. In the language of utility functions, an indifference curve is all combinations of  $X$  and  $Y$  that yield the same level of utility. Suppose we look at the indifference curve that corresponds to 1 unit of utility—

that is, the combinations of bundles for which  $XY = 1$ . Solving this equation for  $Y$ , we have

$$Y = \frac{1}{X}, \tag{A.3.7}$$

which is the indifference curve labeled  $U = 1$  in Figure A.3-5. The indifference curve that corresponds to 2 units of utility is generated by solving  $XY = 2$  to get  $Y = 2/X$  and it is shown by the curve labeled  $U = 2$  in Figure A.3-5. In similar fashion, we generate the indifference curves for  $U = 3$  and  $U = 4$ , which are correspondingly labeled in the diagram. More generally, we get the indifference curve corresponding to a utility level of  $U_0$  by solving  $XY = U_0$  to get  $Y = U_0/X$ .

Consider another illustration, this time with  $U(X, Y) = (\frac{2}{3})X + 2Y$ . The bundles of  $X$  and  $Y$  that yield a utility level of  $U_0$  are again found by solving  $U(X, Y) = U_0$  for  $Y$ . This time we get  $Y = (U_0/2) - (\frac{1}{3})X$ . The indifference curves corresponding to  $U = 1$ ,  $U = 2$ , and  $U = 3$  are shown in Figure A.3-6. Note that they are all linear, which tells us that this particular utility function describes a preference ordering in which  $X$  and  $Y$  are perfect substitutes.

[Figure A.3-6]

### *Using Calculus to Maximize Utility*

Students who have had calculus are able to solve the consumer's budget allocation problem without direct recourse to the geometry of indifference maps. Let  $U(X, Y)$  be the consumer's utility function; and suppose  $M$ ,  $P_X$ , and  $P_Y$  denote income, the price of  $X$ , and the price of  $Y$ , respectively. Formally, the consumer's allocation problem can be stated as follows:

$$\begin{aligned} & \text{Maximize } U(X, Y) \text{ subject to } P_X X + P_Y Y = M. \\ & X, Y \end{aligned} \tag{A.3.8}$$

The appearance of the terms  $X$  and  $Y$  below the “maximize” expression indicates that these are the variables whose values the consumer must choose. The price and income values in

## **THE METHOD OF LAGRANGIAN MULTIPLIERS**

As noted earlier, the function  $U(X, Y)$  itself has no maximum; it simply keeps on increasing with increases in  $X$  or  $Y$ . The maximization problem defined in Equation A.3.8 is called a *constrained maximization problem*, which means we want to find the values of  $X$  and  $Y$  that produce the highest value of  $U$  *subject to the constraint that the consumer spend only as much as his income*. We will examine two different approaches to this problem.

One way of making sure that the budget constraint is satisfied is to use the so-called method of *Lagrangian multipliers*. In this method, we begin by transforming the constrained maximization problem in Equation A.3.8 into the following unconstrained maximization problem:

$$\begin{aligned} \text{Maximize } \mathcal{L} &= U(X, Y) - \lambda (P_X X + P_Y Y - M). \\ X, Y, \lambda \end{aligned} \tag{A.3.9}$$

The term  $\lambda$  is called a Lagrangian multiplier, and its role is to assure that the budget constraint is satisfied. (How it does this will become clear in a moment.) The first-order conditions for a maximum of  $\mathcal{L}$  are obtained by taking the first partial derivatives of  $\mathcal{L}$  with respect to  $X$ ,  $Y$ , and  $\lambda$  and setting them equal to zero:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial U}{\partial X} - P_X = 0, \tag{A.3.10}$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{\partial U}{\partial Y} - P_Y = 0, \tag{A.3.11}$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - P_X X - P_Y Y = 0. \tag{A.3.12}$$

The next step is to solve Equations A.3.10-A.3.12 for  $X$ ,  $Y$ , and  $\lambda$ . The solutions for  $X$  and  $Y$  are the only ones we really care about here. The role of the equilibrium value of  $\lambda$  is to guarantee that the budget constraint is satisfied. Note in Equation A.3.12 that setting the first partial derivative of  $\mathcal{L}$  with respect to  $\lambda$  equal to zero guarantees this result.

Specific solutions for the utility-maximizing values of  $X$  and  $Y$  require a specific functional form for the utility function. We will work through an illustrative example in a moment. But first note that an interesting characteristic of the optimal  $X$  and  $Y$  values can be obtained by dividing Equation A.3.10 by Equation A.3.11 to get

$$\frac{\frac{\partial U}{\partial X}}{\frac{\partial U}{\partial Y}} = \frac{P_X}{P_Y} = \frac{P_X}{P_Y} \tag{A.3.13}$$

Equation A.3.13 is the utility function analog to Equation A.3.8 from the text, which says that the optimal values of  $X$  and  $Y$  must satisfy  $MRS = P_X/P_Y$ . The terms  $\frac{\partial U}{\partial X}$  and  $\frac{\partial U}{\partial Y}$  from Equation A.3.13 are called the *marginal utility of X* and the *marginal utility of Y*, respectively. In words, the marginal utility of a good is the extra utility obtained per additional unit of the good consumed. Equation A.3.13 tells us that the ratio of these marginal utilities is simply the marginal rate of substitution of  $Y$  for  $X$ .

If we rearrange Equation A.3.13 in the form

$$\frac{\partial U / \partial X}{P_X} = \frac{\partial U / \partial Y}{P_Y} \quad (\text{A.3.14})$$

another interesting property of the optimal values of  $X$  and  $Y$  emerges. In words, the left-hand side of Equation A.3.14 may be interpreted as the extra utility gained from the last dollar spent on  $X$ . Similarly, the right-hand side of the equation is the extra utility gained from the last dollar spent on  $Y$ . It is easy to see intuitively why, for the optimal values of  $X$  and  $Y$ , the extra utility gained from the last dollar spent on each must be the same. Suppose, to the contrary, that the extra utility gained from the last dollar spent on  $Y$  exceeded the extra utility from the last dollar spent on  $X$ . The consumer could then spend a dollar less on  $X$  and a dollar more on  $Y$  and end up with more utility than he had under the original allocation. The conclusion is that the original allocation could not have been optimal. Only when the extra utility gained from the last dollar spent on each good is the same will it not be possible to carry out a similar utility-augmenting reallocation.

**An Example.** To illustrate the Lagrangian method, suppose that  $U(X, Y) = XY$  and that  $M = 40$ ,  $P_X = 4$ , and  $P_Y = 2$ . Our constrained maximization problem would then be written as

$$\begin{aligned} \text{Maximize } \mathcal{L} &= XY - \lambda (4X + 2Y - 40). & (\text{A.3.15}) \\ X, Y, \lambda & \end{aligned}$$

The first-order conditions for a maximum of  $\mathcal{L}$  are given by

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial (XY)}{\partial X} - 4\lambda = Y - 4\lambda = 0, \quad (\text{A.3.16})$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{\partial (XY)}{\partial Y} - 2\lambda = X - 2\lambda = 0, \quad (\text{A.3.17})$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 40 - 4X - 2Y = 0. \quad (\text{A.3.18})$$

Dividing Equation A.3.16 by Equation A.3.17 and solving for  $Y$ , we get  $Y = 2X$ ; substituting this result into Equation A.3.18 and solving for  $X$ , we get  $X = 5$ , which in turn yields  $Y = 2X = 10$ . Thus  $(5, 10)$  is the utility-maximizing bundle.<sup>2</sup>

## AN ALTERNATIVE METHOD

There is an alternative way of making sure that the budget constraint is satisfied, one that involves less cumbersome notation than the Lagrangian approach. In this alternative method, we simply solve the budget constraint for  $Y$  in terms of  $X$  and substitute the result wherever  $Y$  appears in the utility function. Utility then becomes a function of  $X$  alone, and we can *maximize* it by taking its first derivative with respect to  $X$  and equating that to

zero.<sup>3</sup> The value of  $X$  that solves that equation is the optimal value of  $X$ , which can then be substituted back into the budget constraint to find the optimal value of  $Y$ .

[Figure A.3-7]

To illustrate, again suppose that  $U(X, Y) = XY$ , with  $M = 40$ ,  $P_X = 4$ , and  $P_Y = 2$ . The budget constraint is then  $4X + 2Y = 40$ , which solves for  $Y = 20 - 2X$ . Substituting this expression back into the utility function, we have  $U(X, Y) = X(20 - 2X) = 20X - 2X^2$ . Taking the first derivative of  $U$  with respect to  $X$  and equating the result to zero, we have

$$\frac{dU}{dX} = 20 - 4X = 0, \quad (\text{A.3.19})$$

which solves for  $X = 5$ . Plugging this value of  $X$  back into the budget constraint, we discover that the optimal value of  $Y$  is 10. So the optimal bundle is again (5, 10), just as we found using the Lagrangian approach. For these optimal values of  $X$  and  $Y$ , the consumer will obtain  $(5)(10) = 50$  units of utility.

Both algebraic approaches to the budget allocation problem yield precisely the same result as the graphical approach described in the text. Note in Figure A.3-7 that the  $U = 50$  indifference curve is tangent to the budget constraint at the bundle (5, 10).

## A SIMPLIFYING TECHNIQUE

Suppose our constrained maximization problem is of the general form

$$\begin{aligned} &\text{Maximize } U(X, Y) \text{ subject to } P_X X + P_Y Y = M. \\ &X, Y \end{aligned} \quad (\text{A.3.20})$$

If  $(X^*, Y^*)$  is the optimum bundle for this maximization problem, then we know it will also be the optimum bundle for the utility function  $V[U(X, Y)]$ , where  $V$  is any increasing function.<sup>4</sup> This property often enables us to transform a computationally difficult maximization problem into a simple one. By way of illustration, consider the following example:

<sup>2</sup>Here, the second-order condition for a local maximum is that  $d^2U/dX^2 < 0$ .

$$\begin{aligned} &\text{Maximize } X^{1/3}Y^{2/3} \text{ subject to } 4X + 2Y = 24. \\ &X, Y \end{aligned} \quad (\text{A.3.21})$$

First note what happens when we proceed with the untransformed utility function given in Equation A.3.21. Solving the budget constraint for  $Y = 12 - 2X$  and substituting back into the utility function, we have  $U = X^{1/3}(12 - 2X)^{2/3}$ . Calculating  $dU/dX$  is a bit tedious in this case, but if we carry out each step carefully we get the following first-order condition:

$$\frac{dU}{dX} = (1/3)X^{-2/3}(12 - 2X)^{2/3} + X^{1/3}(2/3)(12 - 2X)^{-1/3}(-2) = 0, \quad (\text{A.3.22})$$

which, after a little more tedious rearrangement, solves for  $X = 2$ . And from the budget constraint we then get  $Y = 8$ .

Now suppose we transform the utility function by taking its logarithm:

$$V = \ln[U(X, Y)] = \ln(X^{1/3}Y^{2/3}) = (1/3)\ln X + (2/3)\ln Y. \quad (\text{A.3.23})$$

Since the logarithm is an increasing function, when we maximize  $V$  subject to the budget constraint, we will get the same answer we got using  $U$ . The advantage of the logarithmic transformation here is that the derivative of  $V$  is much easier to calculate than the derivative of  $U$ . Again, solving the budget constraint for  $Y = 12 - 2X$  and substituting the result into  $V$ , we have  $V = (1/3)\ln X + (2/3)\ln(12 - 2X)$ . This time the first-order condition follows almost without effort:

$$\frac{dV}{dX} = \frac{1/3}{X} - \frac{2(2/3)}{12 - 2X} = 0, \quad (\text{A.3.24})$$

which solves easily for  $X = 2$ . Plugging  $X = 2$  back into the budget constraint, we again get  $Y = 8$ .

The best transformation to make will naturally depend on the particular utility function you start with. The logarithmic transformation greatly simplified matters in the example above, but will not necessarily be helpful for other forms of  $U$ .

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<sup>4</sup>Again, an increasing function is one for which  $V(X_1) > V(X_2)$  whenever  $X_1 > X_2$ .

## Problems

1. Tom spends all his \$100 weekly income on two goods,  $X$  and  $Y$ . His utility function is given by  $U(X, Y) = XY$ . If  $P_X = 4$  and  $P_Y = 10$ , how much of each good should he buy?
2. Same as Problem 1, except now Tom's utility function is given by  $U(X, Y) = X^{1/2}Y^{1/2}$
3. Note the relationship between your answers in Problems 1 and 2. What accounts for this relationship?
4. Sue consumes only two goods, food and clothing. The marginal utility of the last dollar she spends on food is 12, and the marginal utility of the last dollar she spends on clothing is 9. The price of food is \$1.20/unit, and the price of clothing is \$0.90/unit. Is Sue maximizing her utility?

5. Albert has a weekly allowance of \$17, all of which he spends on used CDs ( $C$ ) and movie rentals ( $M$ ), whose respective prices are \$4 and \$3. His utility from these purchases is given by  $U(C) + V(M)$ . If the values of  $U(C)$  and  $V(M)$  are as shown in the table, is Albert a utility maximizer if he buys 2 CDs and rents 3 movies each week? If not, how should he reallocate his allowance?

$C$	$U(C)$	$M$	$V(M)$
0	0	0	0
1	12	1	21
2	20	2	33
3	24	3	39
4	28	4	42