CHAPTER 10
APPENDIX

MATHEMATICAL EXTENSIONS OF THE THEORY OF COSTS

This appendix discusses two extensions of the cost concepts developed in Chapter 10.

The Relationship Between Long-Run and Short-Run Cost Curves

Let us consider first in greater detail the relationship between long- and short-run total costs. Recall that the LTC curve is generated by plotting the *Q* value for a given isoquant against the corresponding total cost level for the isocost line tangent to that isoquant. Thus, for example, in Figure A.10-1, $Q = 1$ is associated with a long-run total cost of LTC_1, $Q = 2$ with LTC_2, and so on.

When K is variable, as it and all other factors are in the long run, the expansion path is given by the line 0*E*. Now suppose, however, that *K* is fixed at K_2^*, the level that is optimal for the production of $Q = 2$. The short-run expansion path will then be the horizontal line through the point (0, K_2^*), which includes the input bundles *X*, *T*, and *Z*. The short-run total cost of producing a given level of output—say, $Q = 1$—is simply the total cost associated with the isocost line that passes through the intersection of the short-run expansion path and the $Q = 1$ isoquant (point *X* in Figure A.10-1), namely, STC_1.

Note in Figure A.10-1 that short- and long-run total costs take the same value for $Q = 2$, the output level for which the short- and long-run expansion paths cross. For all other output levels, the isocost line that passes through the intersection of the corresponding isoquant and the short-run expansion path will lie above the isocost line that is tangent to the isoquant. Thus, for all output levels other than $Q = 2$, short-run total cost will be higher than long-run total cost.

[Figure A.10-1]

The short- and long-run total cost curves that correspond to the isoquant map of Figure A.10-1 are shown in Figure A.10-2. Note in Figure A.10-1 that the closer output is to $Q = 2$, the smaller the difference will be between long-run and short-run total cost. This property is reflected in Figure A.10-2 by the fact that the STC curve is tangent to the LTC curve at $Q = 2$: The closer *Q* is to 2, the closer STC_Q is to STC_2. Note also in Figure A.10-2 that the STC curve intersects the vertical axis at rK_2^*, the fixed cost associated with K_2^* units of *K*.

The production process whose isoquant map is shown in Figure A.10-1 happens to be one with constant returns to scale. Accordingly, its long-run average and marginal cost curves will be the same horizontal line. The position of this line is determined by the slope of the LTC curve in Figure A.10-2. The associated SAC curve will be U-shaped and tangent to the LAC curve at $Q = 2$, as shown in Figure A.10-3.

[Figure A.10-2]

[Figure A.10-3]

There are short-run cost curves not just for $K = K_2^*$, but for every other level of the fixed input as well. For example, the short-run average cost curve when K *is* fixed at K_3^* (the optimal amount of K for producing Q = 3) is shown in Figure A.10-3 as the curve labeled $\mathrm{SAC}_{K=K_3^*}$. Like the SAC curve associated with K_2^*, it too is U-shaped, and is tangent to the LAC curve at Q = 3. The SAC curves will in general be U-shaped and tangent to the LAC curve at the output level for which the level of the fixed input happens to be optimal.

A similar relationship exists in the case of production processes that give rise to U-shaped LAC curves. For such a process, the LAC curve and three of its associated SAC curves are shown in Figure A.10-4. When the LAC curve is U-shaped, note that the tangencies between it and the associated SAC curves do not in general occur at the minimum points on the SAC curves. The lone exception is the SAC curve that is tangent to the minimum point of the U-shaped LAC (SAC_2 in Figure A.10-4). On the downward-sloping portion of the LAC curve, the tangencies will lie to the left of the minimum points of the corresponding SAC curves; and on the upward-sloping portion of the LAC curve, the tangencies will lie to the right of the minimum points.

[Figure A.10-4]

In the text, we noted that one way of thinking of the LAC curve is as an "envelope" of all the SAC curves, like the one shown in Figure A.10-4. At the output level at which a given SAC is tangent to the LAC, the long-run marginal cost (LMC) of producing that level of output is the same as the short-run marginal cost (SMC). To see why this is so, recall that the tangency point represents the quantity level that is optimal for the faxed factor level that corresponds to the particular SAC curve. If we change output by a very small amount in the short run—by either increasing or reducing the amount of the variable input—we will end up with an input mix that is only marginally different from the optimal one, and whose cost is therefore approximately the same as that of the optimal mix. Accordingly, for output levels very near the relevant tangency point, SMC and LMC are approximately the same.

Note also in Figure A.10-4 that the SMC curves are always steeper than the LMC curve. The reason is implicit in our discussion of why LMC and SMC are nearly the same in a neighborhood of the tangency points. Starting at a tangency point—say, at Q_1 in Figure A.10-4—suppose we want to produce an extra unit of output in the short run. To do so, we will have to move from an input mix that is optimal to one that contains slightly more L and slightly less K than would be optimal for producing Q_1 + 1 in the long run. So the cost of that extra unit will be higher in the short run than in the long run, which is another way of saying $SMC_{Q1+1} > LMC_{Q1+1}$.

Now suppose that we start at Q_1 and want to produce 1 unit of output less than before. To do so, we will have to move to an input bundle that contains less L and more K than would be optimal for producing Q_1 - 1. In consequence, our cost savings will be smaller in the short run than they would be in the long run, when we are free to adjust both L and K. This tells us that $LMC_{Q1-1} > SMC_{Q1-1}$. To say that LMC exceeds SMC whenever output is less than Q_1, but is less than SMC when output is greater than Q_1, is the same thing as saying that the LMC curve is less steep than the SMC curve at Q_1.

EXERCISE A.10-1

Consider a production function *Q* = *F*(*K*, *L*) for which only two values of *K* are possible. These two values of *K* give rise to the SAC curves shown in the diagram. What is the LAC curve for this firm?

[Figure Ex. A.10-1]

The Calculus Approach to Cost Minimization

Using the Lagrangian technique discussed in the Appendix to Chapter 3, we can show that the equality of MP/*P* ratios (Equation 10.22 in Chapter 10) emerges as a necessary condition for the following cost-minimization problem:

$$\min_{K,\,L} P_K K + P_L L \qquad \text{subject to } F(K, L) = Q_0. \qquad \text{(A.10.1)}$$

To find the values of K and L that minimize costs, we first form the Lagrangian expression:

$$£ = P_K K + P_L L + \lambda[F(K, L) - Q_0]. \qquad \text{(A.10.2)}$$

The first-order conditions for a minimum are given by

$$\frac{\partial £}{\partial K} = P_K + \frac{\lambda \partial F}{\partial K} 0, \qquad \text{(A.10.3)}$$

$$\frac{\partial £}{\partial L} = P_L + \frac{\lambda \partial F}{\partial L} 0, \qquad \text{(A.10.4)}$$

and

$$\frac{\partial £}{\partial \lambda} = F(K,L) - Q_0 = 0 \qquad \text{(A.10.5)}$$

Dividing Equation A.10.3 by Equation A.10.4 and rearranging terms, we have

$$\frac{\partial F / \partial K}{P_K} = \frac{\partial F / \partial L}{P_L}, \qquad \text{(A.10.6)}$$

which is the result of Equation 10.21 in Chapter 10. (As an exercise, derive the same result by finding the first-order conditions for a maximum level of output subject to a cost limit of *C*.)

An alternative to the Lagrangian technique is to solve the production function constraint in Equation A.10.1 for *K* in terms of *L*, then substitute the result back into the expression for total cost. To illustrate this alternative approach, consider the following example.

EXAMPLE A.10-1

For the production function $Q = F(K, L) = \sqrt{K}\sqrt{L}$ *with* $P_K = 4$ *and* $P_L = 2$*, find the values of K and L that minimize the cost of producing 2 units of output.*

Our problem here is to minimize $4K + 2L$ subject to $F(K, L) = \sqrt{K}\sqrt{L} = 2$. Here the production function constraint is $Q = 2 = \sqrt{K}\sqrt{L}$, which yields $K = 4/L$. *So* our problem is to minimize $4(4/L) + 2L$ with respect to *L*. The first-order condition for a minimum is given by

$$\frac{d[(16/L) + 2L]}{dL} = 2 - \frac{16}{L^2} = 0, \qquad \text{(A.10.7)}$$

which yields $L = 2\sqrt{2}$. Substituting back into the production function constraint, we have $K = 4/(2\sqrt{2}) = \sqrt{2}$.

Problems

1. A firm produces output with the production function

$$Q = \sqrt{K}\sqrt{L}$$

where *K* and *L* denote its capital and labor inputs, respectively. If the price of labor is 1 and the price of capital is 4, what quantities of capital and labor should it employ if its goal is to produce 2 units of output?

2. Sketch LTC, LAC, and LMC curves for the production function given in Problem 2. Does this production function have constant, increasing, or decreasing returns to scale?

3. Suppose that a firm has the following production function:

$$Q(K, L) = 2L\sqrt{K}.$$

a. If the price of labor is 2 and the price of capital is 4, what is the optimal ratio of capital to labor?

b. For an output level of $Q = 1000$, how much of each input will be used?

4. A firm with the production function

$$Q(K, L) = 2\sqrt{KL}$$

is currently utilizing 8 units of labor and 2 units of capital. If this is the optimal input mix, and if total costs are equal to 16, what are the prices of capital and labor?

5. For a firm with the production function

$$Q(K, L) = 3 \text{ In } K + 2 \text{ In } L,$$

find the optimal ratio of capital to labor if the price of capital is 4 and the price of labor is 6.

Answer to Exercise

A.10-1. The LAC curve (bottom panel) is the outer envelope of the two SAC curves (top panel).

[FIGURE Ans. A.10-1]