Consumer's Surplus Without Apology

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By ROBERT D. WILLIG*

The purpose of this paper is to settle the controversy surrounding consumer's surplus[1] and, by so doing, to validate its use as a tool of welfare economics. I will show that observed consumer's surplus can be rigorously utilized to estimate the unobservable compensating and equivalent variations—the correct theoretical measures of the welfare impact of changes in prices and income on an individual.

I derive precise upper and lower bounds on the percentage errors of approximating the compensating and equivalent variations with consumer's surplus. These bounds can be explicitly calculated from observable demand data, and it is clear that in most applications the error of approximation will be very small. In fact, the error will often be overshadowed by the errors involved in estimating the demand curve. The results in no way depend upon arguments about the constancy of the marginal utility of income.

Consequently, this paper supplies specific empirical criteria which can replace the apologetic caveats frequently employed by those who presently apply consumer's surplus. Moreover, the results imply that consumer's surplus is usually a very good approximation to the appropriate welfare measures.

To preview, below I establish the validity of these rules of thumb: For a single[2] price change, if $|\bar{\eta}A/2m^0| \leq .05$, $|\underline{\eta}A/2m^0| \leq .05$, and if $|A/m^0| \leq .9$, then

$$(1) \qquad \frac{\underline{\eta}\,|A|}{2m^0} \leq \frac{C-A}{|A|} \leq \frac{\bar{\eta}\,|A|}{2m^0}$$

and

$$(2) \qquad \frac{\underline{\eta}\,|A|}{2m^0} \leq \frac{A-E}{|A|} \leq \frac{\bar{\eta}\,|A|}{2m^0}$$

Here, A = consumer's surplus area under the demand curve and between the two prices (positive for a price increase and negative for a price decrease)
C = compensating variation corresponding to the price change
E = equivalent variation corresponding to the price change
m^0 = consumer's base income
$\bar{\eta}$ and $\underline{\eta}$ = respectively the largest and smallest values of the income

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[1] Throughout, the term consumer's surplus is used to refer to the area to the left of an individual's fixed-income (Marshallian) demand curve and between the relevant price horizontals. The concept of consumer's surplus originated in 1844 (see Jules Dupuit) and has been controversial ever since. Alfred Marshall, who popularized the tool, stipulated that for it to be validly used the marginal utility of money must be constant (Marshall, p. 842 or David Katzner, p. 152). However, Harold Hotelling wrote that consumer's surpluses "give a meaningful measure of social value. This breaks down if the variations under consideration are too large a part of the total economy of the person . . . " (p. 289). John Hicks too, stated only a gentle caution: "In order that the Marshallian measure of consumer's surplus should be a good measure, one thing alone is needful—that the income effect should be small" (p. 177). More recently, though, Paul Samuelson (pp. 194–95) concluded that consumer's surplus is a worse than useless concept (because it confuses), and I.M.D. Little (p. 180) agreed, calling it no more than a "theoretical toy." Nonetheless, theorists and cost-benefit analysts have persisted in their use of the tool. For justification they resort (see E. J. Mishan, pp. 337–38, for example), with no formal theoretical support, to statements similar to those quoted above from Hotelling and Hicks.

[2] While I restrict attention to single price changes here, analogous, but more complex formulae are derived for multiple price changes in my papers (1973a, b).

elasticity of demand in the region under consideration.

The formulae place observable bounds on the percentage errors of approximating the C or E conceptual measures with observable A. For example, if the consumer's measured income elasticity of demand is 0.8 and if the surplus area under the demand curve between the old and new prices is 5 percent of income, then the compensating variation is within 2 percent of the measured consumer's surplus.

The ratio $|A|/m^0$ can be interpreted as a measure of the proportional change in real income due to the price change.[3] In most applications, the ratio will be very small. Measured income elasticities of demand tend to cluster closely about 1.0, with only rare outliers. Thus it can be expected that $\bar{\eta}|A|/2m^0$, the most important of the terms in (1) and (2), will usually be small enough to permit conscious and unapologetic substitution of A for C or E in studies of individual welfare.[4]

Should $\underline{\eta}|A|/2m^0$ be large, A would not be close to C and E. For such rare cases, formulae are provided below in Section IV which enable the estimation of C and E from the observable $\bar{\eta}$, $\underline{\eta}$, m^0, and A.

I. The Compensating and Equivalent Variations

In this section, I present definitions of conceptual tools to measure the costs or benefits of price changes to an individual consumer. While these theoretical measures are not directly observable, the analysis that follows in succeeding sections will show that they can be empirically estimated with consumer's surplus.

Throughout I will be assuming that the consumer behaves as though he were choosing his consumption bundle $X=X^1, X^2, \ldots, X^n$ to maximize an increasing strictly quasi-concave ordinal utility function $U(X)$ subject to the budget constraint $\sum p_i X^i = m$. The resulting demand functions, denoted $X^i(p, m)$, are assumed to be differentiable. The indirect utility function, defined by

$$l(p, m) \equiv U[X^1(p, m), X^2(p, m), \ldots, X^n(p, m)]$$

relates the price and income parameters to the maximum level of utility the consumer can achieve under the resulting budget constraint. Clearly, by nonsatiation, $l(p, m)$ is monotone increasing in income m, and decreasing in prices p.

The indirect utility function can be used to make statements about individual welfare. Let the base, initial situation be characterized by prices p^0 and income m^0, while an alternative situation can be summarized by p', m'. The economic well-being of the consumer in the different situations can be compared by means of the ordinal ranking of the numbers $l(p^0, m^0)$ and $l(p', m')$.

Another way to effect this welfare test is to compare the income change $m'-m^0$ to the smallest income adjustment needed to make the consumer indifferent to the change in prices from p^0 to p'. If $m'-m^0$ is larger, then welfare is greater in the new situation, and inversely.

This test level of income adjustment is called the compensating income variation, denoted by C below. Symbolically,

$$(3) \qquad l(p^0, m^0) = l(p', m^0 + C)$$

The welfare test above

$$(4) \quad l(p', m') \gtreqless l(p^0, m^0) \quad \text{as} \quad m' - m^0 \gtreqless C$$

follows immediately from (3) by nonsatiation. Thus the compensating variation is an individual's cost-benefit concept which makes price changes perfectly commensurable with changes in

[3] Or the ratio can be interpreted using the words of Hotelling quoted in fn. 1 as the relative size of the variation.

[4] Formulae (1) and (2) reflect the cautions (see fn. 1) of both Hotelling and Hicks.

income.

Similarly, the equivalent variation in income (E) can be defined[5] by

$$(5)\qquad l(p^0, m^0 - E) = l(p', m^0)$$

In words, $-E$ is the income change which has the same welfare impact on the consumer in the base situation as have the changes in prices from p^0 to p'. It reduces the impacts of different price changes down to the single dimension of income. As such, the equivalent variation concept can be used to rank the consumer's levels of well-being under various sets of prices. With the definitions $l(p^0, m^0-E')=l(p', m^0)$ and $l(p^0, m^0-E'')=l(p'', m^0)$, these welfare tests, too, follow from nonsatiation:

$$(6)\qquad l(p', m^0) \gtreqless l(p'', m^0) \quad \text{as} \quad E'' \gtreqless E'$$
$$l(p', m^0) \gtreqless l(p^0, m') \quad \text{as} \quad m^0 - E \gtreqless m'$$

The welfare tests (4) and (6) show that the compensating and equivalent variations are cost-benefit concepts which can be used to evaluate the impact of microeconomic policy on an individual.[6] These concepts derive practical importance from the fact that they can be estimated from observable consumer's surplus.

II. Consumer's Surplus

The compensating and equivalent variations can be most incisively studied and related to consumer's surplus by means of the income compensation function.[7] This is denoted by $\mu(p \mid p^0, m^0)$ and is defined to be the least income required by the consumer when he faces prices p to achieve the same utility level he could enjoy (by maximizing behavior) under the parameters p^0, m^0. Thus, by definition,

$$(7)\qquad l[p, \mu(p \mid p^0, m^0)] = l(p^0, m^0)$$

Trivially, we have

$$(8)\qquad \mu(p^0 \mid p^0, m^0) = m^0$$

Now, we can see that the compensating and equivalent variations can be expressed or redefined in terms of the income compensation function. From (3), $m^0+C=\mu(p' \mid p^0, m^0)$, or combining with (8),

$$(9)\qquad C = \mu(p' \mid p^0, m^0) - \mu(p^0 \mid p^0, m^0)$$

Similarly, from (5), $m^0-E=\mu(p^0 \mid p', m^0)$, or

$$(10)\qquad E = \mu(p' \mid p', m^0) - \mu(p^0 \mid p', m^0)$$

These relationships serve as the bridge to consumer's surplus.

It is well known[8] that

$$(11)\qquad \frac{\partial \mu(p \mid p^0, m^0)}{\partial p_i} = X^i(p, \mu(p \mid p^0, m^0))$$

This system of partial differential equations, together with the boundary condition (8), is the heart of analytical welfare economics.[9] The compensating and equivalent variations, or any measure of individual welfare that accepts the individual's own consumption preferences, can be calculated from the complete demand functions via (11) and (8).

Restricting attention to changes in a single price, p_1, let $p^0=(p_1^0, p_2^0, \ldots, p_n^0)$ and $p'=(p_1', p_2^0, \ldots, p_n^0)$. Use the Fundamental Theorem of Calculus and (11) to rewrite (9) and (10) as

[5] The definitions (3) and (5) correspond to those of Hicks, p. 177, and Samuelson, p. 199.

[6] They also can serve as building blocks for methodologies to make social welfare judgments. The Compensation Principle is a well-known example (see Tibor Scitovsky).

[7] This theoretical tool was introduced by Lionel McKenzie, and definitively studied by Leonid Hurwicz and Hirofumi Uzawa.

[8] See Hurwicz and Uzawa for a state-of-the-art derivation. Heuristically, (11) says that the first-order income change, $d\mu$, required to compensate for the price increase, dp_1, is just the augmentation needed to buy the old consumption bundle, $X(p, \mu,(p \mid p^0, m^0))$, at the new prices $p_1+dp_1, p_2, \ldots, p_n$, rather than at the old prices p. The irrelevance to this calculation of the concomitant substitution effects is the result of the envelope theorem.

[9] This point of view was taken by Herbert Mohring.

$$(12)\quad C = \int_{p_1^0}^{p_1'} X^1(p_1, p_2^0, \ldots, p_n^0, \mu(p_1, p_2^0, \ldots, p_n^0 \mid p_1^0, p_2^0, \ldots, p_n^0, m^0))\,dp_1$$

$$(13)\quad E = \int_{p_1^0}^{p_1'} X^1(p_1, p_2^0, \ldots, p_n^0, \mu(p_1, p_2^0, \ldots, p_n^0 \mid p_1', p_2^0, \ldots, p_n^0, m^0))\,dp_1$$

These formulae express the compensating and equivalent variations as areas under demand curves, between the old and new price horizontals. The demand curves are not Marshallian in that the income parameters are not constant. Instead, they are Hicksian compensated demand curves, because the income parameters include compensation which varies with the price to keep the consumer at a constant level of utility. The only distinction between C and E in (12) and (13) is the level of utility the compensation is designed to reach.

Referring to Figure 1, C is the area $p_1^0p_1'be$ under the demand curve compensated to $l(p^0, m^0)$. This curve crosses the Marshallian curve $X^1(p, m^0)$ at p_1^0, since $\mu(p^0 \mid p^0, m^0) = m^0$. With $p_1' > p_1^0$, if X^1 is noninferior ($\partial X^1/\partial m \geq 0$) this compensated curve lies above the Marshallian one for $p_1 > p_1^0$, since $\mu(p_1, p_2^0, \ldots, p_n^2 \mid p^0, m^0) \geq m^0$ whenever $p_1 > p_1^0$. Similarly, E is the area $p_1^0p_1'af$ under the demand curve compensated to $l(p', m^0)$. This Hicksian curve crosses the Marshallian one at p_1', and lies below it for $p_1 < p_1'$. The area usually called consumer's surplus is $p_1^0p_1'ae$, defined by the observable Marshallian demand curve. Denoting this area by A, we have, then, $C \geq A \geq E$, for noninferior X^1 (the inequalities reverse for X^1 inferior). Of course, it also follows immediately that if there is no income effect ($\partial X^1/\partial m \equiv 0$), $C = A = E$.

These qualitative results may be useful for some cost-benefit analyses. For example, suppose a policy would raise both an individual's income and the price of a noninferior good. If the observable con-

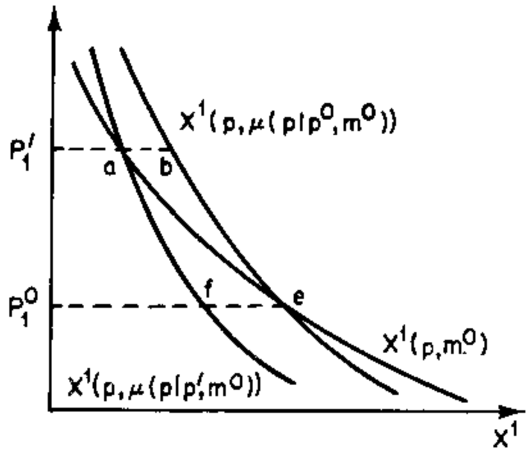

FIGURE 1

sumer's surplus area A were greater than the income boost, it could be inferred from the inequality that C also would be greater. Then, from the welfare test (4), an analyst could conclude that the policy would be injurious to the consumer.

However, usually more information than this is needed about C and E. What is required is a methodology to estimate the welfare measures from observable data. In the next section I show how C and E can be explicitly calculated from observables when the income elasticity of demand is constant.

III. Constant Income Elasticity

Constant income elasticity of demand for X^1 means that

$$\frac{\partial X^1(p, m)}{\partial m} \frac{m}{X^1(p, m)} \equiv \eta$$

Then, we have the simple differential equation $dX^1/X^1 = \eta(dm/m)$ which can be integrated from $X^1(p, m^0)$ to yield

$$X^1(p, m) = X^1(p, m^0)\left[\frac{m}{m^0}\right]^\eta$$

The entire income compensation function can be derived by substituting this expression into (11) and solving the resulting differential equation with boundary condition (8). We have, suppressing unchanging arguments,

$$\frac{d\mu}{dp_1} = X^1(p_1, \mu) = X^1(p_1, m^0)\left[\frac{\mu}{m^0}\right]^\eta$$

or

$$\mu^{-\eta}d\mu = (m^0)^{-\eta}X^1(p_1, m^0)dp_1$$

Then, integration between p_1^0 and p_1', remembering that $\mu(p_1^0)=m^0$, yields

$$(14)\quad \frac{[\mu(p_1')]^{1-\eta} - [m^0]^{1-\eta}}{1-\eta} = (m^0)^{-\eta}\int_{p_1^0}^{p_1'} X^1(p_1, m^0)dp_1$$

for $\eta \neq 1$, and for $\eta = 1$,

$$ln\, \mu(p_1') - ln\, m^0 = \frac{1}{m^0}\int_{p_1^0}^{p_1'} X^1(p_1, m^0)dp_1$$

Hence, after rearranging we have these explicit expressions for the income compensation function:

$$(15)\quad \mu(p_1' \mid p_1^0, m^0) = m^0\left[1 + \left(\frac{1-\eta}{m^0}\right)\int_{p_1^0}^{p_1'} X^1(p_1, m^0)dp_1\right]^{1/1-\eta}$$

$\eta \neq 1$

$$(16)\quad \mu(p_1' \mid p_1^0, m^0) = (m^0)\exp\left[\frac{1}{m^0}\int_{p_1^0}^{p_1'} X^1(p_1, m^0)dp_1\right]$$

$\eta = 1$

These give the welfare measure μ in terms of the potentially observable constant income elasticity of demand and the consumer's surplus area under the Marshallian demand curve. Let us denote this area by

$$(17)\qquad A \equiv \int_{p_1^0}^{p_1'} X^1(p_1, m^0)dp_1$$

From (15), we see that if $\eta=0$, $\mu(p_1' \mid p_1^0, m^0) = m^0 + A$. However, from (16) we see that if preferences are homothetic, the consequent unitary η does not imply any equalities among C, E, and A. Below, for expositional convenience, I ignore the case $\eta = 1$.

Recalling the definitions of C and E, (9) and (10), and loosely applying to (15) this Taylor approximation,

$$(1+t)^{1/1-\eta} \approx 1 + \frac{t}{1-\eta} + \frac{\eta t^2}{2(1-\eta)^2}$$

(where ≈ means "approximately equal to"), we get:

$$C \approx A + \frac{\eta A^2}{2m^0}, \qquad E \approx A - \frac{\eta A^2}{2m^0}$$

$$\frac{C-A}{A} \approx \frac{\eta A}{2m^0} \quad \text{and} \quad \frac{A-E}{A} \approx \frac{\eta A}{2m^0}$$

This was the striking result on the percentage error of approximating C with A which was previewed in the introduction. The next section will establish this formula rigorously for nonconstant income elasticity of demand.

IV. Estimation Results

Assume that in the region of price-income space under consideration,[10] $\bar{\eta}$ and $\underline{\eta}$ are upper and lower bounds, respectively, on $(\partial X^1(p, m)/\partial m)(m/X^1(p, m))$, with neither equal to 1.[11] It follows from the Mean Value Theorem that

$$(18)\qquad \left(\frac{m_2}{m_1}\right)^{\underline{\eta}} \leq \frac{X^1(p, m_2)}{X^1(p, m_1)} \leq \left(\frac{m_2}{m_1}\right)^{\bar{\eta}}$$

for $m_2 \geq m_1$

Let us consider the welfare impact of a price increase from p_1^0 to p_1'. Since $\mu(p_1 \mid p^0, m^0) \geq \mu(p_1^0 \mid p^0, m^0)$ for $p_1 \geq p_1^0$, we can set $m_2 = \mu(p_1)$ and $m_1 = \mu(p_1^0) = m^0$ in (18):

[10] This region is $\{(p, m): p_1 = \alpha p_1^0 + (1-\alpha)p_1', 0 \leq \alpha \leq 1; p_i = p_i^0, i \neq 1; m = \gamma m^0 + (1-\gamma)\mu(p \mid p^0, m^0), 0 \leq \gamma \leq 1;$ and $X^1(p, m) > 0\}$.

[11] Either $\bar{\eta}$ or $\underline{\eta}$ can be arbitrarily close to 1.

$$\left[\frac{\mu(p_1)}{m^0}\right]^{\underline{\eta}} \leq \frac{X^1(p_1,\mu(p_1))}{X^1(p_1, m^0)} \leq \left[\frac{\mu(p_1)}{m^0}\right]^{\bar{\eta}}$$

Rearranging, and substituting from (11) yields

$$0 \leq X^1(p, m^0)^{-\underline{\eta}} \leq \frac{\partial\mu(p)}{\partial p_1}[\mu(p)]^{-\underline{\eta}} = \partial\left[\frac{\mu(p)^{1-\underline{\eta}}}{1-\underline{\eta}}\right] \Big/ \partial p_1$$

and

$$0 \leq \partial\left[\frac{\mu(p)^{1-\bar{\eta}}}{1-\bar{\eta}}\right] \Big/ \partial p_1 = \frac{\partial\mu(p)}{\partial p_1}\mu(p)^{-\bar{\eta}} \leq X^1(p, m^0)(m^0)^{-\bar{\eta}}$$

Integrating these relationships with respect to p_1 between p_1^0 and p_1' (as in (14)) preserves the inequalities. Rearrangement of the resulting relationships yields these bounds:

$$\text{(19)}\quad m^0\left[1+(1-\underline{\eta})\frac{A}{m^0}\right]^{1/1-\underline{\eta}} \leq \mu(p' \mid p^0, m^0) \leq m^0\left[1+(1-\bar{\eta})\frac{A}{m^0}\right]^{1/1-\bar{\eta}}$$

provided $\underline{\eta}, \bar{\eta} \neq 1,\ 1+(1-\underline{\eta})\frac{A}{m^0} > 0$

and $1+(1-\bar{\eta})\frac{A}{m^0} > 0$

For the case of a price decrease from p_1^0 to p_1', since $\mu(p_1 \mid p_1^0, m^0) \leq m^0$ for $p_1 \leq p_1^0$, we can set $m_2 = m^0$ and $m_1 = \mu(p_1)$ in (18), and then follow the same sequence of steps. Once again, (19) emerges, but reference to (17) shows that here A is negative.

Invoking the definition (9), (19) can be rewritten as

$$\text{(20)}\quad \frac{\left[1+(1-\underline{\eta})\frac{A}{m^0}\right]^{1/1-\underline{\eta}} - 1 - \frac{A}{m^0}}{|A|/m^0} \leq \frac{C-A}{|A|} \leq \frac{\left[1+(1-\bar{\eta})\frac{A}{m^0}\right]^{1/1-\bar{\eta}} - 1 - \frac{A}{m^0}}{|A|/m^0}$$

Also, using (10) and reversing the roles of p' and p^0 in (19) (but not in the definition of A) gives

$$\text{(21)}\quad \frac{\left[1-(1-\underline{\eta})\frac{A}{m^0}\right]^{1/1-\underline{\eta}} - 1 + \frac{A}{m^0}}{|A|/m^0} \leq \frac{A-E}{|A|} \leq \frac{\left[1-(1-\bar{\eta})\frac{A}{m^0}\right]^{1/1-\bar{\eta}} - 1 + \frac{A}{m^0}}{|A|/m^0}$$

The measures of a consumer's welfare can be tightly estimated from observables via (19)–(21), regardless of the size of A/m^0, if $1\pm(1-\underline{\eta})A/m^0>0$, $1\pm(1-\bar{\eta})A/m^0>0$, and if $\underline{\eta}$, and $\bar{\eta}$ are sufficiently close in value.[12] Of course, in the limit, as $\underline{\eta}$ approaches $\bar{\eta}$, (19) reduces to the constant elasticity formula (15). Moreover, we shall see that if the absolute values of $\underline{\eta}A/2m^0$ and $\bar{\eta}A/2m^0$ are small, then (20) and (21) reduce to elegant rules of thumb.

Table 1 displays the numerical values of the following coefficients for selected choices of η and a:

[12] The most plausible cause of the negation of these conditions is $(\partial X^1/\partial m)(m/X^1)\to\infty$. However, regions in which X^1 is identically zero can be ignored, since there both μ and A are unchanging. To handle the case in which $X^1=0$ and $\partial X^1/\partial m\neq 0$ near the boundary of the relevant region, bounds on μ can be derived from bounds on $\partial X^1/\partial m$. Because these are generally more gross than (19), the best approach is to take this tack only in the vicinity of the singularity, use (19) on the rest of the path of integration, and splice the sets of inequalities together. The formulae for such procedures can be found in my 1973a, b papers. An explicit solution for μ when $\partial X^1/\partial m$ is independent of m is also reported there.

TABLE 1[a]

η \ a	.001	.005	.010	.020	.030	.040	.050	.075	.100	.150	.200	.250
-2.00	-.001 -.001 -.001	-.005 -.005 -.005	-.010 -.010 -.010	-.020 -.019 -.021	-.030 -.029 -.032	-.040 -.038 -.043	-.050 -.046 -.054	-.075 -.067 -.086	-.100 -.086 -.121	-.150 -.121 -.209	-.200 -.152 -.316	-.250 -.180 -.480
-1.01	-.001 -.001 -.001	-.003 -.003 -.003	-.005 -.005 -.005	-.010 -.010 -.010	-.015 -.015 -.016	-.020 -.019 -.021	-.025 -.024 -.027	-.038 -.035 -.041	-.051 -.046 -.056	-.076 -.066 -.090	-.101 -.085 -.129	-.126 -.102 -.174
.30	.000 .000 .000	.001 .001 .001	.002 .002 .002	.003 .003 .003	.005 .005 .005	.006 .006 .006	.008 .008 .008	.011 .011 .011	.015 .015 .015	.023 .023 .023	.030 .029 .031	.038 .036 .039
.50	.000 .000 .000	.001 .001 .001	.003 .003 .003	.005 .005 .005	.008 .008 .008	.010 .010 .010	.013 .013 .013	.019 .019 .019	.025 .025 .025	.038 .038 .038	.050 .050 .050	.063 .063 .063
.70	.000 .000 .000	.002 .002 .002	.004 .004 .004	.007 .007 .007	.011 .011 .010	.014 .014 .014	.018 .018 .017	.026 .027 .026	.035 .035 .035	.053 .054 .051	.070 .072 .068	.088 .090 .085
.90	.000 .000 .000	.002 .002 .002	.005 .005 .004	.009 .009 .009	.014 .014 .013	.018 .018 .018	.023 .023 .022	.034 .034 .033	.045 .046 .044	.068 .070 .065	.090 .095 .085	.113 .120 .105
1.01	.001 .001 .001	.003 .003 .003	.005 .005 .005	.010 .010 .010	.015 .015 .015	.020 .020 .020	.025 .026 .025	.038 .039 .037	.051 .052 .049	.076 .080 .072	.101 .108 .094	.126 .138 .116
1.10	.001 .001 .001	.003 .003 .003	.006 .006 .006	.011 .011 .011	.017 .017 .016	.022 .022 .022	.028 .028 .027	.041 .043 .040	.055 .057 .053	.083 .088 .078	.110 .119 .102	.138 .152 .125
1.20	.001 .001 .001	.003 .003 .003	.006 .006 .006	.012 .012 .012	.018 .018 .018	.024 .024 .024	.030 .031 .029	.045 .047 .043	.060 .063 .057	.090 .097 .084	.120 .132 .110	.150 .169 .131
1.50	.001 .001 .001	.004 .004 .004	.008 .008 .007	.015 .015 .015	.023 .023 .022	.030 .031 .029	.038 .039 .036	.056 .059 .054	.075 .080 .070	.113 .125 .102	.150 .173 .132	.188 .224 .160
2.00	.001 .001 .001	.005 .005 .005	.010 .010 .010	.020 .020 .020	.030 .031 .029	.040 .042 .038	.050 .053 .048	.075 .081 .070	.100 .111 .091	.150 .176 .130	.200 .250 .167	.250 .333 .200
3.00	.002 .002 .002	.008 .008 .008	.015 .015 .015	.030 .031 .029	.045 .047 .043	.060 .064 .056	.075 .082 .069	.113 .129 .100	.150 .180 .129	.225 .302 .180	.300 .455 .226	.375 .657 .266
5.00	.003 .003 .002	.013 .013 .012	.025 .026 .024	.050 .053 .047	.075 .082 .069	.100 .114 .089	.125 .147 .109	.188 .244 .154	.250 .362 .193	.375 .716 .261	.500 1.477 .317	.625 ** .364
10.00	.005 .005 .005	.025 .026 .024	.050 .053 .047	.100 .115 .089	.150 .186 .126	.200 .271 .160	.250 .374 .191	.375 .774 .257	.500 1.916 .312	.750 ** .396	1.000 ** .460	1.250 ** .509

[a] Each group of three numbers includes, from the top, $\eta a/2$, $[(1+(1-\eta)a)^{1/1-\eta}-1-a]/a$, and $[(1-(1-\eta)a)^{1/1-\eta}-1+a]/a$. The entry ** indicates that $(1+(1-\eta)a)<0$.

$$\frac{\eta a}{2}, \quad \frac{[1+(1-\eta)a]^{1/1-\eta}-1-a}{a}$$

and

$$\frac{[1-(1-\eta)a]^{1/1-\eta}-1+a}{a}$$

The latter two expressions encompass the forms of the bounds in (20) and (21), when a is interpreted as $|A|/m^0$.[13] It can

[13] For example, the value of the lower bound in (20) when $\eta=2$ and $A/m^0=-.05$ is .048. This can be found in Table 1 as the value of $[(1-(1-\eta)a)^{1/1-\eta}-1+a]/|a|$ when $\eta=2$ and $a=.05$.

be readily seen from the table that for the ranges of parameter values studied,[14] when $|\eta a/2|$ is small (say less than .05), $\eta a/2$ is close enough (within .005) to the actual bounds for most practical purposes. This numerical observation corroborates the loose application to (15) of the Taylor Series expansion in Section III. More importantly, it establishes the rules of thumb previewed in (1) and (2).[15]

Addition of (1) and (2) yields a check on the numerical proximity of C and E: when $|\underline{\eta}A/2m^0| \leq .05$, $|\bar{\eta}A/2m^0| \leq .05$, and $|A/m^0| \leq .9$,

$$(23) \qquad \frac{\underline{\eta}|A|}{m^0} \leq \frac{C-E}{|A|} \leq \frac{\bar{\eta}|A|}{m^0}$$

So, the analysis hinges on the magnitudes of η and A/m^0. As discussed in the introduction, in most practical applications $|\eta A/2m^0|$ and $|A/m^0|$ are likely to be small enough for the rules of thumb to apply. If not, equations (19)–(21) and Table 1 will be useful. Even if the calculated error bounds are too large to be ignored, the compensating and equivalent variations may still be usefully estimated from the data via the formulae.

V. Individual Welfare and Consumer's Surplus

With the approximation results in hand, let us return to the question of how to make statements about individual welfare, based on observable data. Remember from (4) that $l(p', m') \gtreqless l(p^0, m^0)$ as $m' - m^0 \gtreqless C$. With the empirical information that $\underline{C} \leq C \leq \overline{C}$, where $\underline{C}$ and $\overline{C}$ can be calculated from (20) or (22), it can be concluded that

$$(24) \qquad \begin{aligned} l(p', m') &> l(p^0, m^0), \quad \text{if } m' - m^0 > \overline{C} \\ l(p', m') &< l(p^0, m^0), \quad \text{if } m' - m^0 < \underline{C} \end{aligned}$$

If $\underline{C}$ and $\overline{C}$ are close in value, (24) provides a welfare test of considerable power.[16] If $|\bar{\eta}A/2m^0|$ and $|\underline{\eta}A/2m^0|$ are small enough, both $\underline{C}$ and $\overline{C}$ can be safely replaced in (24) by A. Otherwise, they can be calculated from $\underline{\eta}$, $\bar{\eta}$, A, and m^0.

To conclude, at the level of the individual consumer, cost-benefit welfare analysis can be performed rigorously and unapologetically by means of consumer's surplus.

[14] These seem to include most values that would be found for these parameters in actual applications.

[15] When $|\underline{\eta}A/2m^0| \leq .05$ and $|\bar{\eta}A/2m^0| \leq .05$, it suffices for $1 \pm (1-\underline{\eta})A/m^0 > 0$ and $1 \pm (1-\bar{\eta})A/m^0 > 0$ that $|A/m^0| < .9$.

[16] Another welfare comparison (which may be useful for an analysis of social welfare with a Bergsonian social welfare function) is made possible by the fact (see Hurwicz and Uzawa) that $\mu(p^0|p, m)$, viewed as a function of p and m, is a proper indirect utility function.

$$\mu(p^0|p, m) = E + m$$

where E is the equivalent variation associated with a change from p^0 to p. Hence this particular ordinal indirect utility function can be exactly expressed by areas under compensated demand curves, as in (13), or it can be estimated from consumer's surplus via (19), (21), or (2).

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