

## Undominated Nash Implementation in Bounded Mechanisms\*

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We study implementation in undominated Nash equilibrium by bounded mechanisms. (An undominated Nash equilibrium is a Nash equilibrium in which no agent uses a weakly dominated strategy. A mechanism is bounded if every dominated strategy is dominated by some undominated strategy.) We identify necessary conditions and sufficient conditions for such implementation. These conditions are satisfied in virtually all economic environments, and are also satisfied by interesting correspondences from the social choice literature. For economic settings, we provide a particularly simple implementing mechanism for which the undominated equilibrium outcomes coincide with those obtained from the iterative elimination of weakly dominated strategies. *Journal of Economic Literature* Classification Numbers: 022, 025, 026. © 1994 Academic Press, Inc.

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## 1. INTRODUCTION

Implementation theory studies the extent to which social goals can be achieved by decentralized decision making procedures. The social goals are given by a *social choice correspondence*. A procedure is represented by a *mechanism*, which specifies the possible actions available to members of a society, as well as the outcomes of these actions. Once an assumption is made about what determines individual or group behavior, the equilibrium outcomes to the mechanism are precisely the social objectives that can be achieved by the decentralized procedure. If these outcomes coincide with the social choice correspondence (SCC), then the SCC is *implementable* in the precise sense that it can be decentralized.

The implementation problem is made non-trivial by two considerations. First, the socially desirable outcome may conflict with the interests of some individual or group of individuals. Second, information relevant to the determination of a socially desirable outcome may not be commonly available. In either case, the decentralized procedure must ensure that individual or group incentives are controlled so that socially desirable outcomes emerge from the social decision making process.

Implementation theory has proceeded along two lines. One branch studies the outcomes of specific mechanisms, such as sequential majority voting or the agenda process, and characterizes what can be decentralized by such procedures. The second branch has been less concerned with the mechanism, and has concentrated on characterizing those social objectives which can be decentralized by *some* mechanism, not necessarily one corresponding to any specific class of decision making procedures.

This second branch has been successful in characterizing which SCCs are implementable using a variety of solution concepts and for various information structures.<sup>1</sup> Much of this work stems from the early contribution of Maskin (1977), who identified an intuitive condition, called monotonicity, which is necessary for Nash implementation when there are

<sup>1</sup> Complete information studies include implementation in Nash equilibrium (Maskin (1977, 1985), Saijo (1988), Moore and Repullo (1990), Dutta and Sen (1991b), Danilov (1992)), subgame perfect equilibrium (Moore and Repullo (1988), Abreu and Sen (1990)), undominated Nash equilibrium (Palfrey and Srivastava (1991)), strong equilibrium (Dutta and Sen (1991a)), undominated equilibrium (Jackson (1992)), backward induction (Herrero and Srivastava (1992)), iterated elimination of dominated strategies (see Moulin (1983), for a summary), virtual implementation (Matsushima (1988), Abreu and Sen (1991), Abreu and Matsushima (1990a, 1992a)), and the early results on dominant strategies (Gibbard (1973), Satterthwaite (1975)). Characterizations when there is incomplete information include Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1987, 1989a), Mookherjee and Reichelstein (1990), and Jackson (1991), all of whom employ Bayesian equilibrium, and Palfrey and Srivastava (1989b), who study undominated Bayesian equilibrium. A recent survey can be found in Palfrey (1992).

no informational asymmetries across agents in society. Monotonicity is satisfied by some normative criteria of interest in specific domains, such as the Pareto correspondence in pure exchange economies. Unfortunately, many important normative criteria, including most criteria in the social choice literature such as the (Condorcet) top cycle set and the uncovered set, fail to satisfy monotonicity, as do other desirable objectives such as the (unconstrained) Walrasian correspondence in exchange economies. This has led to investigations of implementation with stronger equilibrium concepts such as subgame perfect equilibrium and undominated Nash equilibrium.

Our concern here is with undominated Nash implementation. An undominated Nash equilibrium is a Nash equilibrium in which no one uses a weakly dominated strategy. Palfrey and Srivastava (1991), show that with this refinement of Nash equilibrium, *any* social choice correspondence<sup>2</sup> can be implemented. This is a striking result in that it says that for almost any normative criterion, it is possible to construct a decentralizing procedure and control individual incentives to ensure that the outcomes are precisely those prescribed by the criterion. It also implies that there is no conflict between the normative goal of social choice theory (which develops normative criteria) and the positive goal (which examines if these criteria are attainable).

The effect of the assumption that agents do not use dominated strategies is thus quite startling. We generally expect stronger equilibrium concepts to expand the class of implementable SCCs, since in designing an implementing mechanism, it is usually quite easy to obtain a desired outcome as an equilibrium.<sup>3</sup> The difficulty is usually in ruling out other undesired equilibria. Stronger solution concepts tend to be more powerful in ruling out undesired equilibria, hence allowing larger classes of SCCs to be implemented. While we would expect this refinement to expand the class of implementable SCCs relative to Nash implementable SCCs, such a great expansion is surprising.

It turns out that some of the power of this result derives from the fact that we have not imposed any restrictions on the implementing mechanism. In particular, some undesired outcomes are ruled out through infinite strings of dominated actions, where each action dominates a previous action, but where no dominating action is undominated itself. Hence, it is possible that an agent playing the constructed mechanism has no undominated best response to the strategies of the

<sup>2</sup> Their result covers correspondences satisfying no veto power when there are at least three agents in society.

<sup>3</sup> There are some exceptions, such as implementation in "strict" Nash equilibrium or implementation in dominant strategies.

others. For such a mechanism, it is no longer clear that an agent will not play a dominated action, and suggests that it may not be appropriate to apply the concept of undominated Nash equilibrium to such mechanisms. Consequently, it is important to investigate what can be implemented by mechanisms in which every agent always has an undominated best response.

The restriction on mechanisms which prevents the use of an infinite chain of dominated strategies is defined by Jackson (1992). This restriction is called "boundedness," and requires that if an action is weakly dominated, then it is weakly dominated by an undominated action. Jackson provides an example (see Example 1 in Section 3 below) to show that boundedness restricts the set of SCCs which can be implemented in undominated Nash equilibrium.

This brings us to the topic of this paper, which is to characterize the class of SCCs which are implementable in undominated Nash equilibrium by bounded mechanisms. We provide a characterization of SCCs which are implementable in undominated Nash equilibrium by bounded mechanisms. While the restriction to bounded mechanisms eliminates some SCCs which are undominated Nash implementable with unbounded mechanisms, it still admits a large set of implementable SCCs. This set is larger than that for Nash implementation; the "chained" condition we identify can be seen as a weakening of monotonicity. We show that many SCCs of interest, including those identified in the social choice literature and those relating to "economic" settings<sup>4</sup> (including public goods), can be implemented by bounded mechanisms. Finally, we investigate implementation under more stringent requirements on the implementing mechanism, namely, that the mechanism not only be bounded, but also not admit any mixed strategy equilibrium for any von Neumann–Morgenstern representation of the ordinal preferences. We solve this problem for economic settings, but leave as an open question the full characterization for completely general environments. The mechanism constructed for economic environments is simple and intuitive, and is dominance solvable. It also covers the two-agent case.

The next section contains the model and basic definitions. In Section 3, we study in detail the case of strict preferences. In Section 4, we consider the use of mixed strategies and prove a general possibility result for a class of economic environments. Independently, Sjostrom (1994), has obtained results similar to those presented in Section 4 of this paper.

<sup>4</sup> In fact, we work with "separable" environments. An environment is separable if it is possible to "punish" a group of agents without affecting any of the other agents. This is possible in environments such as economic ones where there exists a private good.

## 2. THE MODEL

The set of *alternatives* is denoted by  $A$ . There are a finite number,  $N$ , of *agents*, indexed by  $i = 1, \dots, N$ .  $S$  denotes the set of *states*, where  $s \in S$  summarizes the preferences of all agents. In state  $s$ ,  $R^i(s)$  denotes the complete and transitive *preference relation* of agent  $i$ . We denote by  $P^i(s)$  the strict preference relation corresponding to  $R^i(s)$  and by  $I^i(s)$  the indifference relation corresponding to  $R^i(s)$ . If  $aI^i(s)b$  for all  $a, b \in A$ , then we say that  $i$  is *completely indifferent* at  $s$ . The state is known by each agent.

$F: S \rightarrow 2^A$  is a *social choice correspondence*; for each  $s$ , it selects a subset of  $A$ . This is interpreted as the set of socially desirable outcomes in state  $s$ . We turn next to a formal description of decentralized decision making procedures.

A *mechanism* (or game form) is a pair  $(M, g)$  consisting of a *message space*  $M = M^1 \times M^2 \times \dots \times M^N$  and an *outcome function*  $g: M \rightarrow A$ .  $M^i$  is called the *message space* of  $i$ . An element  $m^i \in M^i$  is referred to as a strategy for  $i$ , and we write  $m \in M$  in the form  $m = (m^1, m^{-i})$ , and call  $m$  a *strategy profile*.

DEFINITION 1. A strategy profile  $m \in M$  is a (pure strategy)<sup>5</sup> *Nash equilibrium* at  $s$  if  $g(m^i, m^{-i})R^i(s)g(n^i, m^{-i})$  for all  $i$  and all  $n^i \in M^i$ .

Given a mechanism  $(M, g)$ , let

$$NE(s) = \{a \in A \mid \exists m \in M \text{ with } g(m) = a \text{ and } m \text{ is a Nash equilibrium at } s\}$$

be the set of Nash equilibrium outcomes at  $s$ .

DEFINITION 2. A strategy  $m^i$  is *weakly dominated* at  $s$  if there exists a strategy  $n^i$  such that  $g(n^i, m^{-i})R^i(s)g(m^i, m^{-i})$  for all  $m^{-i}$  and  $g(n^i, m^{-i})P^i(s)g(m^i, m^{-i})$  for some  $m^{-i}$ .

DEFINITION 3. A strategy profile  $m \in M$  is an *undominated Nash equilibrium* at  $s$  if  $m$  is a Nash equilibrium at  $s$  and for all  $i$ ,  $m^i$  is not weakly dominated at  $s$ .

Given a mechanism  $(M, g)$ , let

$$UNE(s) = \{a \in A \mid \exists m \in M \text{ with } g(m) = a \text{ and } m \text{ is an undominated Nash equilibrium at } s\}$$

be the set of undominated Nash equilibrium outcomes at  $s$ .

DEFINITION 4. A social choice correspondence  $F: S \rightarrow 2^A$  is *undomi-*

<sup>5</sup> A discussion of the role of mixed strategies is given in Section 4.

ated Nash implementable if there exists a mechanism  $(M, g)$  with  $UNE(s) = F(s)$  for all  $s$ .

The following condition was identified by Palfrey and Srivastava (1991), for undominated Nash implementation.

*Property Q.* (Value distinction): For  $s$  and  $s'$ , if  $a \in F(s)$  and  $a \notin F(s')$ , then there exists  $i$  with  $R^i(s) \neq R^i(s')$  and  $i$  is not completely indifferent at  $s'$ .

**DEFINITION 5.**  $F$  satisfies *no veto power* if for all  $s$ , if there exists  $a \in A$  such that for at least  $N - 1$  agents,  $aR^i(s)b$  for all  $b$ , then  $a \in F(s)$ .

**THEOREM** (Palfrey and Srivastava (1991)). *If  $F$  is undominated Nash implementable then  $F$  satisfies Property Q. Further, if  $N \geq 3$  and  $F$  satisfies no veto power and Property Q, then  $F$  is undominated Nash implementable.*

The strength of the above theorem is that it substantially expands the class of implementable SCCs relative to Nash implementation. That is, property *Q* is a much weaker condition than monotonicity, which is necessary for Nash implementation.<sup>6</sup> For example, when  $A$  is a finite set and the set of preferences is the set of all linear orders (strict preferences) then any social choice function satisfying no veto power is undominated Nash implementable (by the above theorem), while only dictatorial social choice functions are Nash implementable (Dasgupta *et al.*, (1979)).

As discussed in the introduction, some of the power of the above result is due to the fact that we have not imposed any restrictions on the mechanism. As shown by Jackson (1992), excluding infinite chains of dominated strategies can restrict the set of implementable SCCs (see Example 1 in the next section). The following definition rules out such constructions.

**DEFINITION 6** (Jackson (1992)). *A mechanism  $(M, g)$  is bounded if for all  $s$ ,  $i$ , and  $m^i$ , if  $m^i$  is weakly dominated at  $s$ , then there exists  $n^i \in M^i$  which weakly dominates  $m^i$  at  $s$  and is not weakly dominated at  $s$ .*

### 3. BOUNDED IMPLEMENTATION IN GENERAL ENVIRONMENTS

#### A. An Example

We begin with an example from Jackson (1992), which not only shows that boundedness restricts the class of implementable SCCs but also motivates the condition we identify.

<sup>6</sup> A SCC is *monotonic* if for every  $s$  and  $s'$ , if  $x \in F(s)$  and  $x \notin F(s')$  then there exists an agent  $i$  and an outcome  $y \in A$  such  $xR^i(s)y$  and  $yP^i(s')x$ .

EXAMPLE 1 (Jackson (1992)).  $S = \{s, s'\}$ ,  $A = \{x, y\}$ ,  $N = 5$ ,  $F(s) = \{x\}$ ,  $F(s') = \{y\}$ , and preference are

$s$					$s'$				
1	2	3	4	5	1	2	3	4	5
y	y	x	x	y	y	y	x	x	x
x	x	y	y	x	x	x	y	y	y

This social choice function satisfies no veto power and  $N \geq 3$ . Therefore, by the theorem of Palfrey and Srivastava (1991), stated in the previous section,  $F$  is undominated Nash implementable. However,  $F$  is not implementable by a bounded mechanism, as we now argue.

In this example only agent 5's preferences change between  $s$  and  $s'$ . Yet, the SCC always picks 5's worst alternative. We can see the role of unbounded mechanisms as follows. Consider a mechanism  $(M, g)$  which implements  $F$  in undominated Nash equilibria. Let  $m$  be a UNE at  $s$ . Then  $g(m) = x$ . Since only 5's preferences change from  $s$  to  $s'$ ,  $m$  is also a Nash equilibrium at  $s'$ . However, since  $x \notin \text{UNE}(s') = F(s')$  we know that  $m^5$  must be dominated at  $s'$  by  $\tilde{m}^5$ .  $\tilde{m}^5$  must have  $g(\tilde{m}^5, m^{-5}) = x$ , and must also be a Nash equilibrium. (Only agents 1 through 4 might deviate and since  $\tilde{m}^5$  dominates  $m^5$  for 5 at  $s'$ ,  $\tilde{m}^5$  must provide outcome  $y$  against a smaller set of actions of agents other than 5 than  $m^5$  does.) Thus in turn  $\tilde{m}^5$  must also be dominated. This leads to an infinite string of actions for agent 5, each dominating a previous one, but none being undominated.

Such a mechanism would clearly not be bounded. The argument against such mechanisms is clear. At  $s'$  agent 5 does not choose any action which provides  $x$  (his desired outcome) against  $m^{-5}$  since all such actions are dominated in an infinite string. Instead, agent 5 chooses an action which provides  $y$  (5's undesired outcome) against  $m^{-5}$ . Against  $m^{-5}$ , it is clearly in agent 5's interest to choose an action which provides  $x$ . However, since each such action is dominated by another, agent 5 cannot decide which one to choose. For such a mechanism it is no longer reasonable to argue that agents will not play a dominated action.

Bounded mechanisms have the property that each agent can always "make up their mind" among a set of actions when considering whether a strategy is weakly dominated. We will also require that every agent always has a best response to the any actions of the other agents. For this class of mechanisms, undominated Nash equilibria are thus an appropriate solution.

*B. The Chained Condition*

The above example illustrates what goes wrong when we try to implement the above SCC by a bounded mechanism. It also provides some insight into the class of SCCs which can be implemented by bounded mechanisms. It turns out that an important part of implementing an SCC by a bounded mechanism is to find an appropriate agent (in place of agent 5) and a  $z_1$  and  $z_2$  as summarized by the following condition. The remainder of this section is written assuming strict preferences. All the definitions and results extend to weak preferences in a natural way. This is done in Jackson *et al.*, (1990).

DEFINITION 7.  $F$  is *chained at*  $x, s, s'$  if there exists an agent,  $i$ , and alternatives,  $y_1$  and  $y_2$ , such that  $y_1 P^i(s) y_2 P^i(s') y_1$  where either:

(A)  $y_1 = x$

or

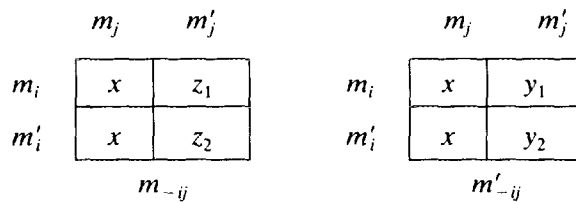
(B) There exist agent  $j \neq i$ , and alternatives  $z_1$  and  $z_2 \notin \{x, z_1\}$  such that:

(B1)  $z_2 P^j(s') x, z_2 P^i(s') z_1$

(B2)  $z_1 = x$  or  $x P^j(s) z_1$ .

$F$  is *chained* if it is chained at  $x, s, s'$  whenever  $x \in F(s) \setminus F(s')$ . The intuition behind the condition is actually quite simple. Part A says that if  $x \in F(s) \setminus F(s')$ , then  $F$  satisfies the standard monotonicity condition relative to  $x, s, s'$ . If not, then there exist a pair of agents who are linked in a special way (part B of the definition). The first agent,  $i$ , has different preferences in states  $s$  and  $s'$ . From Palfrey and Srivastava (1991), we know that this means we can construct a mechanism such that  $x$  is an undominated Nash equilibrium at  $s$ , but not at  $s'$ , by giving  $i$  a strategy that "breaks" any Nash equilibrium producing  $x$  at  $s$  by exploiting weak dominance. In order for the implementing mechanism to be bounded, at least one such dominating strategy must not be weakly dominated. This will be true, if we can find some agent  $j \neq i$  (part B of the chained condition), for whom playing the original strategy is not a best response to this alternative dominating strategy of  $i$ . However, by giving  $j$  a better response, we must not destroy the weak dominance for agent  $i$ . The existence of  $z_1$  and  $z_2$  with the properties in part B of the condition ensure that this can be done.

Part B of the chained condition is illustrated in the following figure





In the figure, the row and column players are the “test” agents in the chained condition. The matrices represent the outcome function at two profiles of messages of all agents other than  $i$  and  $j$ . The outcomes  $x$ ,  $y_1$ ,  $y_2$ ,  $z_1$ , and  $z_2$  correspond to the notation in the chained condition.

Observe that if part B of the condition is satisfied, then  $(m_i, m_j, m_{-ij})$  is a *UNE* at  $s$ , producing outcome  $x$ . At  $s'$ , however,  $m'_i$  weakly dominates  $m_i$ . To ensure that  $(m'_i, m_j, m_{-ij})$ , which also produces outcome  $x$ , is not an equilibrium at  $s'$ , part (B1) requires  $z_2 P^j(s')x$ .

In this manner, part B of the chained condition guarantees that we can eliminate unwanted equilibria by a combination of dominance arguments ( $m'_i$  weakly dominates  $m_i$  at  $s'$  but not at  $s$ ) and best response arguments ( $m'_j$  is a better response to  $m'_i$  than  $m_j$ ). These features are used in the sufficiency proof.

The chained condition is stronger than Property Q in that Property Q requires neither (A) nor (B). It is easy to see that the SSC in Example 1 above is not chained. Nevertheless, the condition is actually very weak; for example, it is always satisfied in the following three general situations.

(i)  $F$  satisfies no veto power and there is a uniformly worst element; i.e., there exists  $w \in A$  such that for all  $s$  and  $i$ ,  $a P^i(s)w$  for all  $a \neq w$ . In this case, if  $x \in F(s) \setminus F(s')$ , no veto power guarantees the existence of  $z_2$  and  $j \neq i$  such that  $z_2 P^j(s')x$ , and the rest of part (B) is satisfied by setting  $z_1 = w$ . Therefore  $F$  is chained.

(ii)  $x \in F(s) \setminus F(s')$  and there exists  $i$  with  $P^i(s) \neq P^i(s')$ ,  $j \neq i$ , and  $y \in A$ , such that both agents prefer  $y$  to  $x$  at  $s'$ . In this case, we set  $z_1 = x$  and  $z_2 = y$ , and  $F$  is chained. This observation will be useful when we analyze economic environments in Section 5.

(iii)  $F$  is monotonic. This guarantees part (A) of the chained condition.

### C. Sufficiency of the Chained Condition

We turn next to sufficiency. We show below that with three or more agents, any chained SCC which satisfies no veto power can be implemented by a bounded mechanism. Before proceeding, we note that we may want to impose restrictions on mechanisms beyond boundedness. We have argued that boundedness is an appropriate restriction given that agents are assumed not to use weakly dominated strategies. Similarly, in order for the “Nash” part of the solution to make sense, we should require that every agent have a best response to every strategy profile of the other agents. We impose this next.

**DEFINITION 8.** A mechanism  $(M, g)$  has the *best response property* if for all  $i$ , for all  $s$ , and for all  $m^{-i}$ , there exists  $m^i \in M^i$  such that  $g(m^i, m^{-i}) R^i(s) g(\bar{m}^i, m^{-i})$  for all  $\bar{m}^i \in M^i$ .

This best response property is not redundant given the boundedness property. It is possible, for instance, to find mechanisms for which all actions are undominated (so the mechanism is bounded) and yet agents' best responses are not always well defined. It is also possible to find mechanisms where best responses are always well-defined, but for which there exists an infinite string of domination, with each strategy in the string dominating the previous one, but no strategy dominating all strategies in the string.

**DEFINITION 9.**  $F$  is boundedly implementable if it is implementable in undominated Nash equilibrium by a bounded mechanism satisfying the best response property.

We remark that an alternative definition of bounded implementation is one that simply requires that all agents have a undominated best response to any strategy of the others. This would be slightly weaker than requiring bounded implementation. However, for the purposes of this paper, this distinction does not make any difference. In particular, Theorems 1, 2, and 3 hold for either definition of bounded implementation.

**THEOREM 1.** *If  $N \geq 3$ ,  $A$  is finite,  $F$  satisfies no veto power, and  $F$  is chained, then  $F$  is boundedly implementable.*

*Proof.* See Appendix.<sup>7</sup> ■

#### D. Weakly Chained: A Necessary Condition

While  $F$  being chained is sufficient for bounded implementation, it is not necessary. The following weaker condition is necessary. For any finite set  $J$ , let  $2^J$  be the set of all subsets of  $J$  (including the empty subset).

**DEFINITION 10.**  $F$  is weakly chained if for all  $s$  and  $s'$ , if  $x \in F(s) \setminus F(s')$ , then either  $F$  is chained at  $x$ ,  $s$ ,  $s'$  or there exists a subset of agents,  $I \subseteq \{1, 2, \dots, N\}$ , with  $\#I > 1$ ,  $j \notin I$ , and a function  $z: 2^I \rightarrow A$  such that

$$\begin{aligned} P^i(s) &\neq P^i(s') && \text{for all } i \in I, \\ z(I)P^j(s')x, \\ xP^j(s)z(\emptyset) &\quad \text{or} \quad x = z(\emptyset), \text{ and} \\ z(C)P^i(s')z(C \setminus \{i\}) &\text{ or } z(C) = z(C \setminus \{i\}) && \text{for all } i \in C \text{ and } C \in 2^I. \end{aligned}$$

**THEOREM 2.** *If  $F$  is boundedly implementable, then  $F$  is weakly chained.*

*Proof.* See Appendix. ■

<sup>7</sup> See Jackson et al., (1990), for a proof with weak preferences.

While this condition is somewhat awkward to express, the intuition is actually straightforward. Note that in comparing  $s$  and  $s'$ , if the only difference between  $s$  and  $s'$  is that the preferences of exactly one agent change, then the restrictions imposed by either "chained" condition are the same. The difference between  $F$  being chained and  $F$  being weakly chained is that in the latter, we might need several agents (as opposed to a single agent) to have weakly dominating strategies before we can find an agent who has a better response. The following example illustrates this point, and also shows that  $F$  being chained is not necessary for bounded implementation.

EXAMPLE 2.  $N = 3, A = \{x, a, b, c, d\}, S = \{s, s'\}$ , and preferences are

$s$			$s'$		
1	2	3	1	2	3
$x$	$x$	$a$	$x$	$x$	$d$
$c$	$c$	$b$	$a$	$b$	$x$
$a$	$b$	$d$	$c$	$c$	$c$
$b$	$a$	$x$	$d$	$d$	$a$
$d$	$d$	$c$	$b$	$a$	$b$

Let  $F(s) = \{x\}, F(s') = \{d\}$ . To see that  $F$  is weakly chained, observe that we can set  $I = \{1, 2\}, j = 3$ , and define the mapping  $z: 2^I \rightarrow A$  by  $z(\emptyset) = c, z(\{1\}) = a, z(\{2\}) = b, z(\{1, 2\}) = d$ . The values of  $z$  correspond to the elements of the matrix on the right of the mechanism which implements  $F$ , where player 1 chooses a column, player 2 a row, and player 3 a matrix.

		1				1	
		$m_1$	$m'_1$			$m_1$	$m'_1$
2	$m_2$	$x$	$x$	2	$m_2$	$c$	$a$
	$m'_2$	$x$	$x$		$m'_2$	$b$	$d$
		$m_3$				$m'_3$	

It can be verified, however, that  $F$  is not chained. Notice that this example can be extended to an environment where preferences are a cross-product,

by letting  $F$  be the social choice correspondence implemented by the above mechanism.

*E. Applications to Voting Rules*

*The Top-Cycle Set*

An interesting social choice correspondence which is not weakly chained and, therefore not boundedly implementable is the Condorcet or top cycle correspondence, defined as follows. For any  $x, y \in A$ , we write  $xD(s)y$  if a strict majority of agents prefer  $x$  to  $y$ .

$$tc(s) = \cap\{B \subseteq A \mid x \in B, y \notin B \text{ implies } xD(s)y\}.$$

The top cycle set at  $s$  is the smallest subset of  $A$  with the property that nothing outside the set is preferred by a strict majority to anything in the set.<sup>8</sup>

EXAMPLE 3.  $N = 3, A = \{x, a, b, c\}, S = \{s, s'\}$ , and preferences are

$s$			$s'$		
1	2	3	1	2	3
$x$	$b$	$c$	$x$	$b$	$c$
$a$	$c$	$a$	$a$	$c$	$a$
$b$	$x$	$b$	$c$	$x$	$b$
$c$	$a$	$x$	$b$	$a$	$x$

Here,  $tc(s) = \{x, a, b, c\}, tc(s') = \{c\}$ , so  $x \in tc(s) \setminus tc(s')$ . Note that only the preferences of agent 1 change between  $s$  and  $s'$ , so  $i = 1$  and chained and weakly chained are equivalent. Consider any choice of  $j, z_1$ , and  $z_2$ .

(1)  $j \neq 1$ : This follows from the fact that  $xP^1(s')y$  for all  $y \neq x$ .

(2) If  $j = 2$ : then, (B2) of the chained condition implies  $z_1 \in \{x, a\}$ . Further,  $z_2P^2(s')x$  implies  $z_2 \in \{b, c\}$ . But  $x$  and  $a$  are preferred by 1 at  $s'$  to both  $b$  and  $c$ , so we cannot satisfy the (B1) requirement,  $z_2P^1(s')z_1$ .

(3) If  $j = 3$ : then (B2) implies  $z_1 = x$ , since  $x$  is 3's worst element at  $s$ . But then (B1) implies  $z_2 = x$  since  $x$  is 1's best element at  $s'$ , and this violates the requirement that  $z_2P^3(s')x$ .

<sup>8</sup> There are several definitions of the top cycle set (see Schwartz (1986)). The one we use corresponds to the GETCHA set.

We conclude that the top cycle correspondence is not weakly chained.<sup>9</sup> This example also shows that the *intersection* of the set of Pareto optimal outcomes and the top cycle set is not boundedly implementable.

### *The Uncovered Set*

An interesting SCC which is chained is the uncovered set, identified by Miller (1977). The uncovered set is a subset of the top cycle set and contains only Pareto optimal alternatives. As shown below, the uncovered set SCC is chained and satisfies no veto power, and therefore is boundedly implementable.

The uncovered set is defined as follows. We say that  $x$  covers  $y$  at  $s$  if  $xD(s)y$  and for all  $z$ ,  $yD(s)z \Rightarrow xD(s)z$ . Thus,  $x$  covers  $y$  if it "defeats"  $y$  and also defeats all alternatives which  $y$  defeats. The *uncovered set* at  $s$ , written  $uc(s)$ , is the set of alternatives which are not covered at  $s$ :

$$uc(s) = \{x \in A \mid x \text{ is not covered at } s\}.$$

In Example 2,  $uc(s) = \{a, b, c\}$  and  $uc(s') = \{c\}$ .

**PROPOSITION 2.** *The uncovered set correspondence is chained.*

*Proof.* Suppose  $x \in uc(s)$  and  $x \notin uc(s')$ . Then, there exists  $z$  such that  $z$  covers  $x$  at  $s'$  but not at  $s$ . There are two cases.

*Case 1: not  $zD(s)x$ .* In this case, since  $zD(s')x$ , there must exist an agent  $i$  such that  $zP^i(s')xP^i(s)z$ . Therefore, the first part of the chained condition is satisfied.

*Case 2: There exists  $y$  such that  $xD(s)y$  while not  $zD(s)y$ .* If it is not the case that  $xD(s')y$ , then there exists an agent  $i$  such that  $yP^i(s')xP^i(s)y$ , so the first part of the chained condition is satisfied. Suppose then that  $xD(s')y$ . Since  $z$  covers  $x$ , we get  $zD(s')y$ . Since we have not  $zD(s)y$ , there exists an agent  $i$  such that  $zP^i(s')yP^i(s)z$ . To complete the argument, we need a  $j \neq i$  such that  $zP^j(s')xP^j(s)y$ . This follows from the facts that  $zD(s')x$  and  $xD(s)y$  and the fact that the  $D$  relation requires that more than  $N/2$  agents prefer an allocation to another. If  $j = i$ , then it follows that  $xP^i(s)zP^i(s')x$  and so the first part of the chained condition is satisfied. Otherwise, let  $z_2 = z$  and  $z_1 = y$  and the second part of the chained condition is satisfied.

<sup>9</sup> This example also shows that the SCC given by  $F(s) = x$ ,  $F(s') = c$  is not chained. We show below that this  $F$  is not boundedly implementable. This shows that bounded implementation differs from implementation via backward induction (Herrero and Srivastava (1992)), since this  $F$  is implementable via backward induction.

**COROLLARY.** *If  $N \geq 3$ , the uncovered SCC is boundedly implementable.*

*Proof:* This follows from Theorem 2 above and the fact that the uncovered set correspondence satisfies no veto power. ■

#### *Plurality Rule*

Another interesting social choice correspondence which is boundedly implementable is the plurality rule. The plurality correspondence is neither Nash implementable nor subgame perfect implementable (see Abreu and Sen (1990)). Since the SCC in Example 2 is not boundedly implementable but is subgame perfect implementable, we see that subgame perfect implementation neither implies nor is implied by bounded implementation. The plurality rule is defined as follows.

For any  $a \in A$  and any  $s \in S$ , let

$$N(a, s) = \#\{i \mid aP^i(s)b \text{ for all } b \in A\},$$

$$F_p(s) = \{a \in A \mid N(a, s) \geq N(b, s) \forall b \in A\}.$$

Then,  $F_p$  is the *plurality correspondence*; at each  $s$ , it picks the best element of the largest group of agents.

**PROPOSITION 3.** *If  $N \geq 3$ , then  $F_p$  is chained.*

*Proof.* Let  $x \in F_p(s) \setminus F_p(s')$ . Then, two cases arise:

- (i)  $x$  is the best element of some  $i$  at  $s$  but not at  $s'$ , or
- (ii) case (i) does not hold and some  $y \in F_p(s')$  is ranked first by more agents at  $s'$  than at  $s$ .

In case (i), there exists  $i$  such that  $xP^i(s)zP^i(s')x$ .

In case (ii), there exists  $i$  such that  $y$  is the best element of  $i$  at  $s'$  and  $y$  is not the best element of  $i$  at  $s$ . Further, since case (i) does not hold we have  $N(y, s') > N(x, s') = N(x, s)$ , so there exists  $j \neq i$  such that  $yP^j(s')x$ . In this case, let  $z_1 = x$ ,  $z_2 = y$ . ■

**COROLLARY.**  *$F_p(s)$  is boundedly implementable if  $I \geq 3$ .*

*Proof.* This follows from Theorem 1 and the fact that with  $N \geq 3$ ,  $F_p$  satisfies no veto power. ■

#### *Borda Count*

The plurality correspondence is a special case of a general class of correspondences called *scoring* correspondences (see Moulin (1983)). While the plurality correspondence is boundedly implementable as shown

above, other scoring correspondences are not. As an example, we consider the scoring correspondence defined by the Borda count. For any  $s$ , let  $B^i(a, s) = k$  if  $a$  is the  $k$ 'th most preferred alternative. Let

$$F_B(s) = \{a \in A \mid \sum_i B^i(a, s) \leq \sum_i B^i(b, s) \forall b \in A\}.$$

Then,  $F_B$  is the Borda correspondence. The following example shows that  $F_B$  is not weakly chained.

EXAMPLE 4.  $F_B$  is not weakly chained.  
 $A = \{a, b, c, d, e\}$ ,  $N = 3$ ,  $S = \{s, s'\}$ , and preferences are

$s$			$s'$		
1	2	3	1	2	3
$a$	$e$	$c$	$a$	$e$	$b$
$b$	$a$	$d$	$b$	$a$	$c$
$c$	$b$	$b$	$c$	$b$	$d$
$d$	$c$	$a$	$d$	$c$	$a$
$e$	$d$	$e$	$e$	$d$	$e$

Here,  $F_B(s) = \{a\}$ ,  $F_B(s') = \{b\}$ .  $F_B$  satisfies no veto power, but it is not weakly chained. To see this, consider outcome  $a$ . The position of  $a$  relative to any other outcome has not changed between  $s$  and  $s'$ , so only the second part of the weakly chained condition must be checked. The only candidate for inclusion in  $I$  is agent 3 (thus weakly chained here is equivalent to chained). The only candidate for  $z(I)$  is  $e$ , and the only candidate for  $j$  is agent 2, since  $a$  is the best element of agent 1 at  $s'$ , and  $e$  is the only alternative which 2 prefers to  $a$  at  $s'$ . But then  $z(\emptyset) = e$  since we must have  $z(I)P^3(s')z(\emptyset)$  or  $z(I) = z(\emptyset)$ . This violates the requirement that  $aP^2(s)z(\emptyset)$  or  $a = z(\emptyset)$ .

We note that in general, Borda winners need not lie in the top cycle set, even if there is a Condorcet winner. This is illustrated here since  $a$  is the Condorcet winner in both states. The example can be modified by adding more alternatives to ensure that the Borda winners lie in the top cycle set at each state, and also to satisfy no veto power.

#### F. Double Implementation

An interesting by-product of Theorem 1 concerns "double" implementation, that is, implementation simultaneously in Nash equilibrium and undominated equilibrium. The following extends results of Yamato (1990), (see also Tatamitani (1991)).

**PROPOSITION 1.** *If  $N \geq 3$ ,  $F$  is monotonic and satisfies no veto power, then there exists a bounded mechanism such that for all  $s$ ,  $\text{UNE}(s) = \text{NE}(s) = F(s)$ .*

*Proof.* The mechanism constructed in our sufficiency proof has this property if  $F$  is monotonic. ■

#### 4. ECONOMIC ENVIRONMENTS

In the previous section, we did not allow for the possibility that agents might use mixed strategies. If agents have von Neumann–Morgenstern utility functions, then the mechanism used to prove Theorem 1 may have mixed strategy equilibria which, with positive probability, lead to outcomes outside of the SCC.

The consideration of mixed strategies leads to several questions which remain unanswered in implementation theory. The mechanism presented in the proof of Theorem 1 can be extended to show that any chained SCC satisfying no veto power can be implemented by a bounded mechanism (i.e., one in which there is no infinite chain of weakly dominated strategies) in which there are no mixed strategy equilibria. This can be done by use of an (infinite) integer game.<sup>10</sup> We do not know if a mechanism can be constructed in which every agent always has a best response to every (mixed) strategy profile of the other agents. This question remains unanswered for Nash implementation as well. The construction of Moore and Repullo (1990), for Nash implementation also involves an integer game for which there are no mixed strategy equilibria. However, there exist mixed strategies of some agents to which other agents have no best response, in which case it seems unreasonable to apply the Nash equilibrium solution concept. The problem that arises in such constructions is that the set of mixed strategies is very large when the mechanism is infinite, and it is difficult to simultaneously have a best response to every mixed strategy profile for every possible von Neumann–Morgenstern representation and at the same time not have any mixed strategy equilibria.

Even if it is possible to construct an implementing mechanism with no mixed strategy equilibrium, it is generally going to be infinite, even in

<sup>10</sup> The restriction to bounded mechanisms does not rule out all types of integer games. Rather, it assures that the process of eliminating dominated strategies is a coherent one. Thus, a restriction to bounded mechanisms may have little to say about whether the Nash solution is appropriate. For example, it is easy to construct bounded mechanisms in which agents do not have a best response to strategy profiles of other agents. In such a mechanism, elimination of dominated strategies may be quite reasonable, but it may be inappropriate to apply the Nash solution concept.



finite environments; an example is given by Jackson (1992). An alternative approach is to examine conditions on SCCs, beyond monotonicity or the chained condition, which characterize Nash or undominated Nash implementation (allowing for mixed strategies) by finite mechanisms in finite environments. Such an approach may be less useful for cases such as pure exchange economies, where it is usually assumed that the set of allocations is infinite. However, the special structure of such environments usually permits particular, simple constructions, as we will show.

The difficulties associated with mixed strategy implementation apply to any Nash-based equilibrium concept, at least in abstract settings. We will show below that in "separable" environments, which include many economically interesting ones, positive results can be obtained with quite simple mechanisms.<sup>11</sup>

Recently, Abreu and Matsushima (1990) have studied implementation by iterated elimination of weakly dominated strategies. They assume that there is a finite set of von Neumann–Morgenstern utility representations, and allow the mechanisms to use random allocations. They show that under some additional (very) weak conditions, any social choice function can be implemented by a finite mechanism. Their construction hinges on their assumption that there is a *finite* set of von Neumann–Morgenstern utility indices. The finiteness restriction and the use of random allocations allows them to construct a mechanism which has no unwanted mixed strategy perfect equilibrium.<sup>12</sup> This still leaves open the central issue concerning mixed strategies: how does one construct a mechanism which is compatible with the solution concept and is immune to mixed strategy equilibria for *every* von Neumann–Morgenstern representation of preferences.

We resolve this question for "separable" environments. We describe a mechanism which implements any social choice function in undominated Nash equilibrium, and has no mixed strategy undominated Nash equilibrium for *any* von Neumann–Morgenstern representation of preferences. The mechanism does not require the use of random allocations.

### *Separable Environments*

We examine bounded implementation in environments which we loosely classify as "separable environments." These environments do not require a finite set of alternatives, and also allow for weak preferences. They

<sup>11</sup> An alternative approach which handles mixed strategies is to consider finite trees of perfect information. There, backward induction can be applied, as studied by Herrero and Srivastava (1992).

<sup>12</sup> There is some concern over the number of steps required to eliminate dominated strategies in the Abreu–Matsushima mechanisms. For discussion of this, see Glazer and Rosenthal (1992), and Abreu and Matsushima (1992b).

include the case of pure exchange economies and all economies with a transferable private good such as public good environments. Here, the chained condition is satisfied by all SCCs.

We will show that in separable environments, bounded implementation can be achieved by a simple mechanism. In fact, any SCC can be boundedly implemented by a mechanism in both pure and mixed strategies. Further, this mechanism is dominance solvable. The result requires the following assumptions, which describe an environment which is separable relative to  $F$ .

(A1) *A worst element relative to  $F$* :  $\exists w \in A$  such that  $aP^i(s)w$  for all  $i, s, s'$ , and  $a \in F(s')$ .

(A2) *Separability*: For all  $a \in A, s' \in S$  and  $J \subseteq N$ , there exists  $a^j \in A$  such that  $a^j I^i(s)w$  for all  $s \in S$  for  $j \in J$ , while  $a^j I^i(s')a$  for  $i \notin J$ .

(A3) *Strict value distinction*: If  $R^i(s) \neq R^i(s')$ , then there exist  $a$  and  $b$  in  $A$  such that  $aP^i(s)b, bP^i(s')a, aP^i(s)w$ , and  $bP^i(s')w$ .

There are many well-studied environments which are separable, as we now discuss.

**EXAMPLE 6 (Transferable Utility).** Here,  $A = B \times C$  where  $C = R^N$  is the set of joint transfers, and  $i$ 's preferences are strictly increasing in the transfer,  $c^i$ . Further, it is assumed that for any  $s \in S, b, b' \in B$ , there exists  $c' \in R^N$  such that  $(b, 0) I^i(s) (b', c')$  for all  $i$ . This says that for any two allocations  $b$  and  $b'$ , there exists a set of transfers such that agents are indifferent between  $b'$  with the transfers and  $b$  without any transfer.

An example of such a setting is in the provision of public goods. Consider the social choice function which selects the efficient decision of whether to undertake a public project, and which distributes costs in proportion to each agent's benefit. Such a social choice function is boundedly implementable (see Jackson and Moulin (1992)). Many other cost sharing rules can also be accommodated.

The case of transferable utility is known to have nice properties relative to implementation. This can be seen in the implementing mechanisms of Moore and Repullo (1988), Glazer and Ma (1989), Jackson and Moulin (1992), and others. The definition of separability, however, admits environments which are not restricted to have quasi-linear utility with transferability. For example, a classical exchange economy is separable with respect to any social choice function which provides each agent with a non-zero allocation.

**EXAMPLE 5 (Pure Exchange Economies).** Here,  $A = \{x \in R_+^{LN} \mid \sum_i x^i \leq e\}$ , where  $e$  is an aggregate endowment and there are  $L$  commodities. Agent  $i$ 's preferences depend only on his own consumption bundle, and is strictly increasing and continuous in this bundle. Here, we can implement any  $F$  which gives a non-zero allocation to each agent. For example,

if each agent has a non-zero endowment, then the Walrasian correspondence is boundedly implementable.<sup>13</sup> Likewise, the no-envy correspondence is boundedly implementable.

**THEOREM 3.** *If the above assumptions are satisfied then any social choice function  $F$  is boundedly implementable. Furthermore, the mechanism can be constructed so that there are no mixed strategy Nash equilibria which only use undominated strategies.*

There are several comments to be made concerning Theorem 3.<sup>14</sup> First, it covers the case of two agents, which is not covered by our other results. So, for instance, it covers bilateral bargaining situations in which transfers are possible. Second, it accounts for mixed strategy equilibria. As discussed by Jackson (1992), the implementation literature has largely ignored the existence of mixed strategy equilibria. Third, it provides the very strong result that *any* SCC can be boundedly implemented. Fourth, the theorem is stated in terms of social choice functions. However, the proof can be modified to also implement correspondences.

*Proof.* Let  $R^i = \{R^i(s) : s \in S\}$  be the set of preferences possible for  $i$ .

$$M^i = [R^i \times R^{i+1}] \cup [A \times A]$$

Agents either announce a pair of preferences (their own and their neighbor's), or they announce a pair of allocations.

$g$  is defined as follows.

*Case 1.* If all agents announce preferences then let

$$J(m) = \{j \mid m_2^j \neq m_1^{j+1}\},$$

and define

$$g(m) = [F(s)]^{J(m)},$$

where  $[\cdot]^{J(m)}$  is as defined in (A2), and  $s$  is the state consistent with the

<sup>13</sup> Note that only the "constrained" Walrasian correspondence is Nash implementable (see Hurwicz *et al.*, (1984)).

<sup>14</sup> Sjostrom (1994), has independently obtained a result very similar to Theorem 3. He investigates a production economy with public goods, which is a fairly general example of a separable environment. Although his mechanism differs from ours, the intuition is roughly the same.

announcements  $m_2$ . [If there is no such state then use a default state  $s' \in S$ .]

*Case 2.* If there exists  $i$  such that  $m^i = (R^i, R^{i+1})$ , while all other agents announce the same pair of allocations  $m^i = (a, b)$ , then

$$g(m) = [a]^{N-i} \quad \text{if } aR^i b \text{ and } aP^i w,$$

while

$$g(m) = [b]^{N-i} \quad \text{if } bP^i a \text{ and } bP^i w,$$

$$g(m) = w \quad \text{otherwise,}$$

where  $[a]^{N-i}$  is defined according to (A2), giving  $a$  to  $i$  and excluding all others.

*Case 3.* All other announcements.

$$g(m) = w.$$

To see that this implements  $F$ :

(1) The only undominated strategies are to announce a pair of preferences, the first of which is the agent's own true preference. [Any announcement concerning the neighbor's preference is undominated.]

It is clear that announcing a pair of allocations is dominated by a truthful revelation of preferences, since announcing allocations always leads to an outcome indifferent to  $w$ .

Next, the announcement of own preferences only affects the outcome under Case 2. Strict value distinction assures that truthful revelation in that case is the only undominated action.

(2) Given (1), the only UNE involves announcing your own true preferences, and your neighbor's true preferences. (This follows since any set of undominated actions must have each agent announcing their own preferences truthfully, and only best response involves matching your neighbor's announcement. The uniqueness of this best response assures that there can be no mixed strategy equilibria.)

(3) From the analysis in (1) above, it is clear that the mechanism is bounded. To see that the mechanism satisfies the best response property, note that by setting  $m_1^i$  to be the true preference, and setting  $m_2^i = m_1^{i+1}$  whenever  $m_1^{i+1}$  is a preference announcement and choosing  $m_2^i$  arbitrarily otherwise, is a best response to the actions of the other agents.

The set of UNE coincide with the strategies left after the iterative

elimination of dominated strategies (or the procedure defined by Borgers (1991)). ■

The mechanism constructed in the proof of Theorem 3 makes heavy use of (A1) and (A2). It “punishes” agents whose messages are inconsistent with the messages of others. The separability assumption (A2) allows the mechanism to punish only certain agents, instead of resorting to severe punishments for all agents when only one has deviated. Abreu and Matsushima (1992a), obtain a similar result with virtual implementation in iteratively undominated strategies. Their result has the advantage of working with smaller punishments, and has the shortcomings of requiring that preferences have a known von Neumann–Morgenstern utility representation in each state, and applying only to finite environments.

## 5. CONCLUDING REMARKS

In summary, this paper presents several results on bounded implementation in undominated Nash equilibrium. First, we identify the *chained* condition which, together with no veto power and at least 3 agents is sufficient for bounded implementation. Second, we identify a slightly weaker necessary condition, called *weakly chained*. Third, we apply these results to show that the uncovered set and plurality rule are boundedly implementable, but the top cycle set and Borda rule are not implementable. Fourth, we identify a domain restriction, *separable environments*, where there exists an implementing mechanism, that is simple and intuitive, applies to the two-agent case, is immune to mixed strategies, is bounded, and is dominance solvable. That domain includes many economic environments of interest.

This is one of the first papers that attempts to characterize what is implementable in general environments under axiomatic restrictions on the mechanism. The requirement of boundedness was motivated by Jackson’s (1992) identification of a conceptual problem with a certain kinds of constructions in the implementation literature. In the course of finding a solution to that problem, it has become evident that there are many other important conceptual issues that remain to be explored in implementation theory. Prominent among these is the problem that many of the mechanisms constructed in the past (including the one in our appendix) may have undesirable mixed strategies in some domains. While the problem can sometimes be overcome in isolation (Moore and Repullo (1990)), or in some kinds of economic environments, its resolution for general

environments remains an open question when coupled with the requirement of boundedness.

## APPENDIX

### *Proof of Theorem 1*

We construct a general implementing mechanism in the following way.

*Preliminary steps in the construction.* The assumption that  $F$  is chained (Definition 7) guarantees that for all  $s, s'$ , and  $x \in F(s) \setminus F(s')$ , there exist  $i, y_1, y_2$  such that  $y_1 P^i(s) y_2 P^i(s') y_1$  and either:

(A)  $y_1 = x$  and  $y_2 P^i(s') x$  or

(B)  $y_1 P^i(s) y_2$  and there exists  $j \neq i$  and there exist allocations  $z_1, z_2$  such that  $z_2 P^j(s') x P^j(s) z_1$  (or  $x = z_1$ ) and  $z_2 R^j(s') z_1$

In case (A) we define two functions,  $i_0(x, s, s')$  and  $y_0(x, s, s')$  such that  $y_0(x, s, s') P^{i_0(s')} x P^{i_0(s)} y_0(x, s, s')$  for  $i = i_0(x, s, s')$ .

In case (B), define the functions  $I(x, s, s'), J(x, s, s'), y_1(x, s, s'), y_2(x, s, s'), z_1(x, s, s'), z_2(x, s, s')$  such that, for  $i = I(x, s, s')$  and  $j = J(x, s, s')$ :

(1)  $y_1(x, s, s') P^i(s) y_2(x, s, s') P^i(s') y_1(x, s, s')$ .

(2)  $z_2(x, s, s') P^j(s') x P^j(s) z_1(x, s, s')$  (or  $x = z_1$ )

(3)  $z_2(x, s, s') P^j(s') z_1(x, s, s')$

Since no one is indifferent over  $A$  at any state, define, for each  $i$ , the functions  $\bar{a}^i(s)$  and  $\underline{a}^i(s)$  such that  $\bar{a}^i(s) P^i(s) \underline{a}^i(s)$  for all  $s$ .

The message space for each agent  $i$  is

$$M^i = A \times S \times S \times \{-(N+3), -(N+2), \dots, -1, 0, 1, 2, \dots, N\}.$$

It is required that the message  $m^i = (x, s, s', 0)$  can only be sent by agent  $I(x, s, s')$ . Also,  $m^i \in F(m^i_2)$  is required for all  $i$ .

Define the outcome function by partitioning the message space as follows.

$$D_1 = \{m \mid \exists s, x \in F(s) \text{ s.t. } \forall k, m^k = (x, s, \cdot, 0)\}$$

$$g(m) = x$$

$$D_2 = \{m \mid \exists s, s' \text{ and } x \in F(s) \text{ s.t. } m^j = (x, s, s', -1) \text{ for } j = J(x, s, s') \text{ and } m^k = (x, s, \cdot, 0) \text{ for all } k \neq J(x, s, s')\}$$

$$g(m) = z_1(x, s, s') \text{ if } m^i_{3^{I(x,s,s')}} = s \text{ or if } m^i_{3^{I(x,s,s')}} = s''$$

$$\text{such that } z_1(x, s, s') P^{I(x,s,s')(s'')} z_2(x, s, s')$$

$$g(m) = z_2(x, s, s') \text{ otherwise.}$$

$D_3 = \{m \mid \exists s, s', x \in F(s), i \text{ s.t. } m^k = (x, s, \cdot, 0), \forall k \neq i_0(x, s, s')\}$   
 and  $m^i = (x, s, s', -2)$  for  $i = i_0(x, s, s')$

$$g(m) = y_0(x, s, s')$$

$D_4 = \{m \mid m \notin D^1 \cup D^2 \cup D^3 \text{ and } \exists s, x \in F(s), \text{ and } i \text{ s.t. } m^k = (x, s, \cdot, 0) \forall k \neq i\}$

$$g(m) = x$$

$D_5 = \{m \mid \exists s, s', x \in F(s) \text{ s.t. } m^k = (x, s, s', -3) \forall k \neq I(x, s, s')\}$   
 and  $m^{I(x,s,s')} = (x, s, \cdot, 0)$

$$g(m) = y_1(x, s, s') \text{ if } m_3^{I(x,s,s')} = s \text{ or if } m_3^{I(x,s,s')} = s'' \text{ such that } y_1(x, s, s')P^{I(x,s,s')}(s'')y_2(x, s, s')$$

$$g(m) = y_2(x, s, s') \text{ otherwise.}$$

$D_6\{m \mid \exists s, s' \text{ (possibly } s = s'), x \in F(s) \text{ and } i \text{ s.t. } m^k = (x, s, s', -(i + 3)) \forall k \neq i\}$

$$g(m) = \bar{a}^i(s') \text{ if } m^i = (x, s, \cdot, 0)$$

$$g(m) = \underline{a}^i(s') \text{ otherwise.}$$

$D_7 = \{\text{all other } m\}$

$$g(m) = m_1^{i^*} \text{ where } i^* \text{ is determined as follows:}$$

$$\text{For each } i, \text{ let } n^i = \max\{0, m_4^i\} \text{ and let } L = n^1 + \dots + n^N$$

$$\text{Then define } i^* = 1 + \text{mod}_N(L)$$

*Claim 1.* At  $s, m^k = (x, s, s, 0)$  for all  $k$  is a Nash equilibrium.

*Proof.* A deviation by agent  $i$  puts the action into one of  $D_2, D_3$ , and  $D_4$ . A move to  $D_4$  changes nothing. From the definition of  $z_i(\cdot)$ , no agent can benefit from moving to  $D_2$ . Since  $xR^i(s)y_0(x, s, s')$  for  $i = i_0(x, s, s')$ , a deviation to  $D_3$  does not improve the payoff to  $i_0(x, s, s')$ .

*Claim 2.* At  $s, m^k = (x, s, s, 0)$  is undominated for each  $k$ .

*Proof.* Region  $D_6$  guarantees that the only strategies that could weakly dominate  $m^k$  at  $s$  are of the form,  $(x, s, s', 0)$ , with  $s \neq s'$ . Such a message is only permitted by  $I(x, s, s')$  or  $i_0(x, s, s')$ . Since  $y_1(x, s, s')P^i(s)y_2(x, s, s')$  for  $i = I(x, s, s')$ , Region  $D_5$  guarantees that  $(x, s, s, 0)$  is not dominated at  $s$  by  $(x, s, s', 0)$ . If  $i = i_0(x, s, s')$  then  $(x, s, s, 0)$  always does at least as well as  $(x, s, s', 0)$ .

*Claim 3.* If  $m$  is a Nash equilibrium at  $s'$  and  $m \notin D_1$  then  $g(m)$  is the best element for at least  $N - 1$  agents and so by no veto power,  $g(m) \in F(s')$ .

*Proof.* From every joint message,  $m$ , in a region other than  $D_1$ , at least  $(N - 1)$  players, each have a unilateral deviation, say  $n^i$ , such that  $(m^{-i}, n^i) \in D_7$  and  $g(m^{-i}, n^i)$  is  $i$ 's best element at  $s$ .

*Claim 4.* At  $s'$ , if  $i = I(x, s, s')$ , then  $m^i = (x, s, s', 0)$  dominates  $m^i = (x, s, s, 0)$ .

*Proof.* In region  $D_5$ ,  $m^i = (x, s, s', 0)$  is strictly better than  $m^i = (x, s, s, 0)$  for  $i = I(x, s, s')$  at  $s'$ . In every other region,  $i$  is weakly better off at  $s'$  with  $m^i = (x, s, s', 0)$  compared to  $m^i = (x, s, s, 0)$ .

*Claim 5.* At  $s'$ , either  $m^i = (x, s, s', 0)$  weakly dominates  $m^i = (x, s, s'', 0)$  for  $i = I(x, s, s')$  or  $z_2(x, s, s')P^i(s'')z_1(x, s, s')$ .

*Proof.* In every region  $i$  is either strictly better off or equally well off reporting  $(x, s, s', 0)$  compared to  $(x, s, s'', 0)$ . If  $(x, s, s', 0)$  does not dominate  $(x, s, s'', 0)$ , then they must lead to the same outcomes in  $D_2$  which implies that  $z_2(x, s, s')P^i(s'')z_1(x, s, s')$ .

*Claim 6.* If  $m \in D_1$  is a UNE at  $s'$ , then  $g(m) \in F(s')$ .

*Proof.* Suppose not. Then  $g(m) \notin F(s')$ . This implies that  $m_1^i = x$  and  $m_2^i = s \neq s'$  for all  $i$ . Since  $F$  is chained, there exists either  $i_0(x, s, s')$  or  $I(x, s, s')$ . In the first case,  $i_0(x, s, s')$  can strictly improve his payoff by reporting  $(x, s, s', -2)$ , which contradicts  $m$  being a Nash equilibrium. In the second case, Claim 4 implies that  $(x, s, s', 0)$  dominates  $(x, s, s, 0)$  for  $I(x, s, s')$ . Since  $m$  is undominated. Claim 5 implies that  $I(x, s, s')$  must be reporting  $(x, s, s', 0)$  or some  $(x, s, s'', 0)$  such that  $z_2(x, s, s')P^i(s'')z_1(x, s, s')$ . But this implies that  $J(x, s, s')$  can strictly improve his payoff by reporting  $(x, s, s', -1)$ , which moves the message from  $D_1$  to  $D_2$  and changes the outcome from  $x$  to  $z_2(x, s, s')$ . This contradicts  $m$  being a Nash equilibrium.

*Claim 7.* The above mechanism implements  $F$  in UNE.

*Proof.* From Claims 1 and 2, for every  $s$  and for every  $x \in F(s)$ , it is an undominated Nash equilibrium for everyone to report  $(x, s, s, 0)$  at  $s$ . From Claim 6 there are no UNE outcomes in  $D_1$  that lie outside  $F(s)$ . From Claim 3, every Nash equilibrium at  $s$  that lies outside of  $D_1$  produces an outcome in  $F(s)$ . Therefore every undominated Nash equilibrium outside of  $D_1$  produces an outcome in  $F(s)$ .

*Claim 8.* The mechanism is bounded and has the best response property.

*Proof.* Since  $A$  is finite, it follows that  $S$  and  $M$  are finite. Any mechanism with a finite message space is bounded and has the best response property. ■

### *Proof of Theorem 2*

Suppose  $F$  is boundedly implementable by  $(M, g)$ ,  $x \in F(s) \setminus F(s')$ , and  $m$  is an undominated Nash equilibrium at  $s$  with  $g(m) = x$ . Then, either  $m$  is not a Nash equilibrium at  $s'$  or  $m$  is weakly dominated at  $s'$ .



If  $m$  is not a Nash equilibrium, then there exists an agent  $i$  and an alternative strategy  $n^i$  with  $x \neq y = g(n^i, m^{-i})$ , and  $xP^i(s)yP^i(s')x$ . In this case (A) is satisfied and  $F$  is chained at  $(x, s, s')$ .

If  $m$  is a Nash equilibrium at  $s'$ , then  $m^i$  must be weakly dominated at  $s'$  for some  $i$ . Let  $I$  be the set of all  $i \in \{1, 2, \dots, N\}$  for whom  $m^i$  is weakly dominated at  $s'$ . For each  $i \in I$ , let  $n^i$  be a dominating strategy. Since  $(M, g)$  is bounded, we can assume that  $n^i$  is itself not weakly dominated at  $s'$  for every  $i \in I$  (otherwise we can simply replace  $n^i$  with an undominated strategy which weakly dominates  $m^i$  at  $s'$ ). Further, it must be the case that  $P^i(s) \neq P^i(s')$  for all  $i \in I$ . Otherwise  $m$  would be weakly dominated at  $s$ .

Next, strict preferences imply that  $g(n^i, m^{-i}) = x$ , for all  $i \in I(s, s')$  (otherwise  $m$  would not be a Nash equilibrium at  $s'$ ). Two cases now arise: (1)  $\#I = 1$ , and (2)  $\#I > 1$ .

*Case (1):  $I = \{i\}$ .* Here,  $g(n^i, m^{-i}) = x$  as argued above, and neither  $n^i$  nor any of the  $m^k$  for  $k \neq i$  are weakly dominated at  $s'$ . Since  $x \notin F(s')$ ,  $(n^i, m^{-i})$  is not a Nash equilibrium at  $s'$ . Hence there exists an agent, say  $j(\neq i)$ , and a strategy for  $j$ , say  $n^j$ , such that  $z_2 = g(n^j, n^i, m^{-i})$  is better than  $x$  for  $j$  at  $s'$ , i.e.  $z_2P^j(s')x$ . Now, let  $z_1 = g(m^j, n^i, m^{-i})$ , so that the situation is given by

		$i$	
		$m^i$	$n^i$
$j$	$m^j$	$x$	$x$
	$n^j$	$z_1$	$z_2$

The requirements of the condition are now clear. Since  $m^j$  is a best response for  $j$  at  $s$ , it must be the case that either  $z_1 = x$  or  $xP^j(s)z_1$ . Since  $n^i$  weakly dominates  $m^i$  at  $s'$ , it must be the case that either  $z_2 = z_1$  or  $z_2P^j(s')z_1$ . If  $z_2 = z_1$  then (A) is satisfied for agent  $j$ . If  $z_2P^j(s')z_1$  then (B) is satisfied. Therefore  $F$  is weakly chained (at  $(x, s, s')$ ).

*Case (2):  $\#I > 1$ .* Suppose, for simplicity, that  $I = \{i, i'\}$ . Since  $g(n^i, m^{-i}) = x$  and  $g(n^{i'}, m^{-i'}) = x$ , the above matrix takes the form

		$i$	
		$m^i$	$n^i$
$i'$	$m^{i'}$	$x$	$x$
	$n^{i'}$	$x$	$y$

Suppose that  $y \neq x$ . Then, it must be the case that  $yP^i(s')x, yP^{i'}(s')x$ , so (B) is satisfied with  $z_2 = y, z_1 = x$  and  $F$  is weakly chained (at  $(x, s, s')$ ).

Suppose then that  $y = x$ , in which case  $g(n^i, n^{i'}, m^{-ii'}) = x$ , and no one is using a weakly dominated strategy. Since  $x \notin F(s')$ , this cannot be a Nash equilibrium. Hence there exists another agent, say  $j$ , and a strategy,  $n^j$ , such that  $n^j$  is a better response for  $j$  to  $(n^i, n^{i'}, m^{-ii'})$  than  $m^i$ . The situation now looks as follows, in which  $j$  chooses the matrix.

		$i$			
		$m^i$	$n^i$		
$i'$	$m^{i'}$	$x$	$x$	$i'$	$m^{i'}$
	$n^{i'}$	$x$	$x$		
		$m^j$			

		$i$			
		$m^i$	$n^i$		
$i'$	$m^{i'}$	$z_\emptyset$	$z_i$	$i'$	$m^{i'}$
	$n^{i'}$	$z_{i'}$	$z_{ii'}$		
		$n^j$			

It must be the case that  $z_{ii'}P^j(s')x$  and  $xR^j(s)z_\emptyset$ . Further, since  $n^i$  weakly dominates  $m^i$  for  $i$ , we must have  $z_iR^i(s')z_\emptyset, z_{ii'}R^i(s')z_{i'}$ , and similarly for agent  $i'$ . Letting  $z(\emptyset) = z_\emptyset, z(\{i, i'\}) = z_{ii'}, z(\{i\}) = z_i, z(\{i'\}) = z_{i'}$ , we get the requisite mapping.

The extension of the above argument when  $I$  consists of more than two agents is straightforward. ■

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