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CONTINUITY AND COMPLETENESS UNDER RISK*

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Continuity and Completeness under Risk*

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Let X be a finite set, and for $m = |X|$, let $\mathcal{P} = \left\{p \in \mathbf{R}_+^m : \sum_i p_i = 1\right\}$ be the set of lotteries over X. Let $\succeq$ be a transitive and reflexive binary relation on $\mathcal{P}$. As usual define $s \succ t$ if $s \succeq t$ and not $t \succeq s$, and $s \sim t$ if $s \succeq t$ and $t \succeq s$. We say that $\succeq$ is non trivial if there exist s and t in $\mathcal{P}$ such that $s \succ t$. The relation $\succeq$ satisfies:
Independence, if for all $p, q, r \in \mathcal{P}$ and $\lambda \in (0,1)$, $p \succeq q$ if and only if $\lambda p + (1-\lambda) r \succeq \lambda q + (1-\lambda) r$;
Herstein Milnor, if for all $p, q, r \in \mathcal{P}$ the set $\{\alpha \in [0,1] : \alpha p + (1-\alpha) q \succeq r\}$ is closed;[1]
Archimedean, if for all $p, q, r \in \mathcal{P}$, $p \succ q$ implies $\lambda p + (1-\lambda) r \succ q$ for some $\lambda \in (0,1)$;
Completeness, if for all p and q, either $p \succeq q$ or $q \succeq p$.

In this note I prove the following Theorem.

Theorem 1 *Suppose $\succeq$ is a transitive, reflexive, non-trivial binary relation on $\mathcal{P}$, that satisfies Independence. If $\succeq$ satisfies any two of the following axioms, it satisfies the third: Herstein-Milnor, Archimedean and Completeness.*

Schmeidler (1971) proved an analogous theorem for the case in which $\succeq$ is a preference relation on a set, not necessarily involving lotteries. He proved that if $\succeq$ on a connected topological set Z is such that for some x and y, $x \succ y$, then closed weak upper and lower contour sets and open strict upper and lower contour sets imply completeness.

That Completeness and Independence imply that HM Continuity and Archimedean are equivalent is trivial and was first claimed by Aumann (1962, p. 453). Karni (2007) proved that under a property weaker than Independence (Local Mixture Dominance), Completeness and Archimedean imply HM Continuity. Hence, we only need to prove that under the assumptions of the Theorem, HM Continuity and Archimedean imply Completeness; to do so I will prove the following Lemma, which together with Schmeidler's theorem will establish the desired result.

Lemma 1 *Suppose X is finite, that $\succeq$ is a transitive, reflexive binary relation on the space $\mathcal{P}$ of lotteries over X, and that $\succeq$ satisfies Independence.*

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[1] The proof below shows that under Independence, this version of HM implies the stronger version which also requires that $\{\alpha : r \succeq \alpha p + (1-\alpha) q\}$ is closed. A similar argument applies to the definition of the Archimedean axiom.

a) If $\succeq$ satisfies HM Continuity then for all p, $\{q : q \succeq p\}$ and $\{q : p \succeq q\}$ are closed.

b) If $\succeq$ satisfies the Archimedean Axiom, then for all p, $\{q : q \succ p\}$ and $\{q : p \succ q\}$ are open in the relative topology in $\mathcal{P}$.

A version of part (a) of the Lemma was established in Proposition 1 in Dubra et al. (2004), but with slightly different axioms: a weaker Independence, and a stronger continuity: Double Mixture Continuity: for any p, q, r, s in $\mathcal{P}$ the following set is closed

$$T = \{\lambda \in [0,1] : \lambda p + (1-\lambda) r \succeq \lambda q + (1-\lambda) s\}.$$

Part (a) of the Lemma is relevant, despite Proposition 1 in Dubra et al., because Double Mixture Continuity is not a standard axiom, and the Independence axiom in this paper is standard. Also, the proof is similar, but simpler. To the best of my knowledge, part (b) is new. Both (a) and (b) could be proved in a more cumbersome manner by appealing to the well known equivalence between algebraic closedness (HM Continuity) and topological closedness (and similarly for openness).

Proof. Proof of (a). In order to show that for all v the set $S = \{r : r \succeq v\}$ is closed take any q in its boundary. If S is a singleton, there is nothing to prove, and if it is not, by the Independence axiom it is a convex set and therefore has a nonempty relative interior. Pick any p in the relative interior of S.

Let B be the open unit ball in the linear space generated by $S - v$, endowed with the relative topology. Fix any $\lambda \in (0,1)$ and any $\varepsilon > 0$. For any $b \in B$, pick $\delta > 0$ small enough that $\varepsilon b + \delta B \subset \varepsilon B$. Since q is in the boundary of S, there exists $w \in S$ such that $\|w - q\| < \delta$, which implies $\varepsilon b + (1-\lambda)(q - w) \in \varepsilon B$ and therefore

$$\begin{aligned} \lambda p + (1-\lambda) q + \varepsilon b &= \lambda p + (1-\lambda) w + (1-\lambda)(q-w) + \varepsilon b \\ &\in \lambda p + (1-\lambda) S + \varepsilon B. \end{aligned} \tag{1}$$

For fixed λ, since p is in the relative interior of S, there exists $\varepsilon > 0$ small enough such that $p + \frac{\varepsilon}{\lambda} B \subseteq S$. Since equation (1) was true for all ε, we obtain

$$\lambda p + (1-\lambda) q + \varepsilon B \subseteq \lambda \left(p + \frac{\varepsilon}{\lambda} B\right) + (1-\lambda) S \subseteq \lambda S + (1-\lambda) S = S.$$

Then, since $0 \in B$, we get that for all $\lambda \in (0,1)$, $\lambda p + (1-\lambda) q \in S$, and by HM, $q \in S$, as was to be shown.

Consider now lower contour sets: for some fixed v, let $T = \{r : v \succeq r\}$. By Independence, for $u = \left(\frac{1}{m}, ..., \frac{1}{m}\right)$, $v \succeq r$ if and only if $v - r \in \{\lambda (p - u) : p \succeq u, \lambda > 0\}$. Hence

$$T = \{r : v \succeq r\} = \{v - m(p-u) : p \succeq u\} \cap \mathcal{P},$$

and closedness follows by closedness of $\{p : p \succeq u\}$.

Proof of (b). We will show that for all p, $\{q : q \succ p\}$ and $\{q : p \succ q\}$ are relatively open in $\mathcal{P}$ by showing that $D = \{\lambda (r - u) : \lambda > 0 \text{ and } r \succ u\}$ is relatively open in the linear space generated by $\mathcal{P} - u$, which we denote $A \equiv aff(\mathcal{P} - u)$. Openness of D implies openness of $\{q : q \succ p\} = (p + D) \cap \mathcal{P}$ (where the equality follows by Independence). Openness of D also implies openness of $\{q : p \succ q\} = (p - D) \cap \mathcal{P}$.

It is easy to see that $A = \{\mu \in \mathbf{R}^m : \sum_1^m \mu_i = 0\}$. Also, given any $\mu \in A$, one can pick λ large enough so as to make $\max_{i \leq m} |\mu_i| < \frac{\lambda}{m}$. Define then $p_i = \frac{\mu_i}{\lambda} + \frac{1}{m}$; it is then straightforward to check that $p \in \mathcal{P}$ and that $\mu = \lambda (p - u)$, showing $A = \{\lambda (p - u) : \lambda > 0, p \in \mathcal{P}\}$.

To show that D is relatively open, pick any $\sigma \in D$ and let $\sigma = \lambda' (p' - u) \in D$. For small enough α, $p \equiv \alpha p' + (1 - \alpha) u$ is in the relative interior of $\mathcal{P}$ and by Independence, $p' \succ u$ ensures that $p \succ u$. Let $\lambda = \frac{\lambda'}{\alpha}$ and note that

$$\sigma = \lambda' (p' - u) = \frac{\lambda'}{\alpha} \alpha (p' - u) = \lambda \alpha (p' - u) = \lambda (p - u).$$

For $i = 1, ..., m-1$, let $c_i = e_i - e_m$ be a basis for A, and for $i = m, ..., 2m-2$, let $c_i = -c_{i-m+1}$. Since p is in the interior of $\mathcal{P}$, for small enough α_i, $p + \alpha_i c_i \in \mathcal{P}$, and since $p \succ u$, the Archimedean axiom ensures that there is some β_i such that $p + \beta_i \alpha_i c_i = (1 - \beta_i) p + \beta_i (p + \alpha_i c_i) \succ u$. Using Independence again, we obtain that for $\gamma_i \equiv \beta_i \alpha_i$ and $\gamma \equiv \min_i \gamma_i > 0$,

$$p + \gamma c_i = \frac{\gamma}{\gamma_i} (p + \gamma_i c_i) + \left(1 - \frac{\gamma}{\gamma_i}\right) p \succ u \tag{2}$$

for all i. Let H denote the convex hull of $\{\gamma c_i\}_{i=1}^{2m-2}$, and note that since for each i, H contains both c_i and $-c_i$,

$$\text{if } \pi \in H \text{ and } \delta < 1 \text{ then } \delta \pi \in H. \tag{3}$$

Moreover, by (2) and Independence, for any s,

$$s \in p + H \Rightarrow s \succ u. \tag{4}$$

For any $x, y \in \mathbf{R}^m$ let $d(x, y) = \max_i |x_i - y_i|$, and let $B = \left\{\mu \in A : d(\mu, 0) < \frac{\gamma}{m-1}\right\}$ be a relatively open set in A. Notice that for any $\mu \in B$, since $\mu = \sum_{i=1}^{m-1} \mu_i c_i$ and $|\mu_i| < \frac{\gamma}{m-1}$, we obtain that for $\delta \equiv \sum_{i=1}^{m-1} \frac{|\mu_i|}{\gamma} < 1$, $\sum_{i=1}^{m-1} \frac{|\mu_i|}{\gamma\delta} = 1$ and therefore

$$\mu = \sum_{i \leq m-1 : \mu_i > 0} \mu_i c_i + \sum_{i \leq m-1 : \mu_i < 0} (-\mu_i)(-c_i) = \delta \left[\sum_{i : \mu_i > 0} \frac{\mu_i}{\gamma\delta} \gamma c_i + \sum_{i : \mu_i < 0} \frac{-\mu_i}{\gamma\delta} (-\gamma c_i)\right]$$

showing that $\mu = \delta \pi$ for π a convex combination of $\{\gamma c_i\}_{i=1}^{2m-2}$, and $\delta < 1$. By equation (3), μ is in H.

For any $\sigma' \in \sigma + \lambda B = \lambda (p - u) + \lambda B$, we have that for some $\mu \in B \subseteq H$, $u + \frac{\sigma'}{\lambda} = p + \mu \in p + B \subseteq p + H$. By equation (4) we obtain $u + \frac{\sigma'}{\lambda} \succ u$ which implies $\sigma' \in D$, which shows that D is relatively open since for any $\sigma \in D$ we have found an open set $\sigma + \lambda B$ containing σ such that $\sigma + \lambda B \subset D$. ■

References

Aumann, R. (1962), "Utility theory without the completeness axiom," *Econometrica* 30(3).

Dubra, J., F. Maccheroni and E. Ok (2004) "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory* 115(1).

Karni, E. (2007), "Archimedean and Continuity," *Mathematical Social Sciences* 53(3).

Schmeidler, D. (1971), "A Condition for the Completeness of Partial Preference Relations," *Econometrica* 39(2).