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A theory of time preferences over risky outcomes

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Abstract

This paper provides two representation theorems for time preferences. They both cover as special cases a variety of time preference models considered in the experimental and theoretical literatures on intertemporal choice. In particular, similarity relations on time and outcomes, exponential, quasi-hyperbolic and hyperbolic discounting are special cases of the theorems. This approach identifies certain factors that are common to time preference structures which look so different.

The paper builds on the recent work by Masatlioglu and Ok [Masatlioglu, Y., Ok, E., 2008. A theory of (relative) discounting. Journal of Economic Theory, in press] on Euclidean bundles and obtains similar representation theorems for the case of compact, separable and connected spaces of bundles. My work allows for the inclusion of the case in which bundles are lotteries.

JEL classification: D11; D90

Keywords: Time preferences; Intransitive preference relations; Similarity relations; Hyperbolic discounting; Time consistency; Stationarity; Present bias

1. Introduction

The theory of time preferences has recently been growing steadily with the contributions of two literatures. One of the lines of research has documented various types of anomalies that the standard theory of discounting fails to encompass. The better known of these anomalies is the time preference reversal: people tend to choose, for example, US$ 100 now and not US$ 110 tomorrow, and US$ 110 in 31 days and not US$ 100 in 30 days. Also relevant to the current paper are the observed violations of transitivity that arise due to the passage of time. The second line of research concerning time preferences is concerned with producing models that would fare better than the discounting model in relation to one specific type of anomaly. Within this literature, one can find, among others, the works of Phelps and Pollak (1968), Laibson (1997) and Rubinstein (2003).

Until very recently there had been no attempts to produce a theory of time preferences that encompassed several of the observed violations. The work of Masatlioglu and Ok (2008) is the first attempt to produce a time preference model which:

(a) Encompasses the following anomalies: hyperbolic discounting, subadditive discounting and intransitivity of dynamic preferences (Roelofsma and Read, 2000). This point is important since, as has been argued

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by Frederick et al. (2002) who reviewed this literature, one can not tell which model is more appropriate according to the experimental evidence. One must therefore allow for various kinds of time preferences.
(b) Encompasses the models of: time discounting (Samuelson, 1937; Koopmans, 1960; Fishburn and Rubinstein, 1982), $\beta - \delta$ or quasi-hyperbolic discounting (Phelps and Pollak, 1968; Laibson, 1997), more general hyperbolic discounting and time preferences based on similarity relations (Rubinstein, 1988, 2003). This point is important since producing a model which encompasses all these different models allows us to distinguish which features are common to all.
(c) Is suitable for applications.
(d) Is axiomatic.

The work of Masatlioglu and Ok (2008) considers only those time preferences that are defined on the prize-time space, just as in Fishburn and Rubinstein (1982). Individuals must choose between dated bundles: bundle x in period t versus bundle y in period s. That is, neither paper considers preferences over streams of consumption bundles.

In this paper I address two shortcomings of the work of Masatlioglu and Ok (2008) while also maintaining literals a-d above. The two drawbacks I deal with in this paper are: their bundles are elements of open boxes of $\mathbf{R}^n$, and they can not cope with the case where bundles are lotteries. I manage to obtain a representation similar to theirs for the case where the bundle space is a compact and connected separable space. The main advantage of extending their analysis to this realm is that my representation theorem can be used to study choices between lotteries. This is important since most economic problems involve some form of uncertainty or risk. Moreover, even though most "real" economic situations involving unknown prizes do not have known probabilities (i.e. there is uncertainty as opposed to risk), the study of preferences over "risky prizes" -time space are relevant. First, representation theorems for this context can be viewed as a first step towards the more "relevant" Savage-type results, since the methods I use are quite simple. Second, most economic models in applied work involve von Neumann–Morgenstern preferences, and not those of Savage or Anscombe-Aumann. The reason for this is that they are simpler to work with. In this sense then, preferences over the "risky prize" -time space can be viewed as a tool for more applied researchers.

This paper is *not* a generalization of the work of Masatlioglu and Ok. Rather, I use a similar set of axioms, to obtain a similar result, in a different context. Some of my axioms are weaker, but I use a transitivity axiom that is stronger than theirs. Yet, my transitivity axiom is still (a lot) weaker than standard transitivity. Finally, in this paper I do not discuss uniqueness, which was dealt with by Masatlioglu and Ok (2008), or with applications of the model, since my model can also be used in all of their settings.

2. The model and axioms

Let X be a compact, separable connected topological space. The set of times is an interval T, containing 0, in $\mathbf{R}_+$.

Let $D \equiv X \times T$ be the space of dated bundles. For each $t \in T$, by the t th time projection of $\succsim \subset D \times D$, I mean the binary relation $\succsim_t$ on X defined as $x \succsim_t y$ iff $(x, t) \succsim (y, t)$.

The following definition introduces the main objects of the present analysis.

Definitions. A binary relation $\succsim$ on D is said to be a time preference on D if it satisfies the following conditions:

(i) $\succsim$ is complete, i.e. $\forall (x, t), (y, s) \in D$, $(x, t) \succcurlyeq (y, s)$ or $(y, s) \succcurlyeq (x, t)$.
(ii) $\succsim$ is continuous: Upper and Lower contour sets of $\succcurlyeq$ are closed.[1]
(iii) $\succsim_0$ is transitive.
(iv) $\succsim_0 = \succsim_t$ for all $t \in T$.

[1] This is, for all $(x, t) \in D$, the sets $U = \{(y, s) : (y, s) \succcurlyeq (x, t)\}$ and $L = \{(y, s) : (x, t) \succcurlyeq (y, s)\}$ are closed. Under completeness, this notion of continuity is equivalent to the stronger version which I will use throughout the paper: $\succcurlyeq$ is a closed subset of $D \times D$.

All axioms are imposed on $\succsim$, a time preference on $D \equiv X \times T$

(A1) Time Discounting (TD). For any $x, y, z \in X$ and $s, t, v \in T$, if $v \leq t$ then

$$(x, t) \left\{ \begin{matrix} \succ \\ \succsim \end{matrix} \right\} (y, s) \quad \text{implies} \quad (x, v) \left\{ \begin{matrix} \succ \\ \succsim \end{matrix} \right\} (y, s).$$

This axiom is standard, and just says that the passage of time is bad, or that bundles are good. The following axiom is a "natural" complement to the previous one, and says that if for a given bundle one time period (later, say) is worse than another, the same is true for all bundles: the effect of time does not depend on the specific bundle.

(A2) Orthogonality (O). For all $x, y \in X$ and $s, t \in T$

$$(x, t) \succcurlyeq (x, s) \quad \text{,if and only if} \quad (y, t) \succcurlyeq (y, s).$$

If $(x, t) \sim (x, s)$ for some $x \in X$, and $t, s \in T$, we say that s and t are *equivalent*. Given Orthogonality, if we say that s and t are equivalent, that does not depend on the choice of x used for comparison of the time periods. The following axiom is the main axiom of this paper. It will allow for the separation of the effects of time and bundles in the preferences.

(A3) Irrelevance (I). For all $w, x, y, z \in X$ and $r, s, t, v \in T,\ r < s, t < v$:

$$\left. \begin{matrix} (y, r) \sim (z, s) \\ \text{and} \\ (y, t) \sim (z, v) \end{matrix} \right\} \Rightarrow (w, r) \succcurlyeq (x, s) \Leftrightarrow (w, t) \succcurlyeq (x, v)$$

The reason why the axiom "should" be true, is that when it is true that $(y, r) \sim (z, s)$ and $(y, t) \sim (z, v)$, the individual deems the passage of time from r to s as "equivalent" to the passage of time from t to v. Then, when comparing (w, t) and (x, v), the individual can refer to his choices between (w, r) and (x, s). It is a stability property of preferences: when we change the time periods, preferences don't change. Irrelevance is equivalent to the Separability axiom of Masatlioglu and Ok when $\succcurlyeq_0 = \succcurlyeq_t$ for all t (which they also assume). In their formulation the conclusion of the axiom is $(w, r) \sim (x, s) \Leftrightarrow (w, t) \sim (x, v)$. The axiom is similar to hexagon the conditions usually used for the separation of preferences in the time-prize spaces, or for additive representations (see for example Debreu, 1960; Karni and Safra, 1998; Fishburn and Rubinstein, 1982; Wakker, 1988, 1989). The Irrelevance axiom for time preferences is weaker (in the presence of Orthogonality), for example, than the formulation of Fishburn and Rubinstein: for all $x, y \in X$ and all $s, t, s+\tau, t+\tau \in T$,

$$(x, t) \sim (y, t+\tau) \quad \text{implies} \quad (x, s) \sim (y, s+\tau).$$

To see so, recall that in the presence of condition $\succcurlyeq_0 = \succcurlyeq_t$ for all t of time preferences, the Irrelevance axiom is equivalent to: for all $w, x, y, z \in X$ and $r, s, t, v \in T,\ r < s, t < v$:

$$\left. \begin{matrix} (y, r) \sim (z, s) \\ \text{and} \\ (y, t) \sim (z, v) \end{matrix} \right\} \Rightarrow (w, r) \sim (x, s) \Leftrightarrow (w, t) \sim (x, v).$$

Then, by the Fishburn Rubinstein condition we have $(y, r) \sim (z, s) \Rightarrow (y, t) \sim (z, t+s-r)$ and then $(y, t) \sim (z, v)$ implies that $(z, t+s-r) \sim (z, v)$, so that $t+s-r$ is equivalent to v. Then, $(w, r) \sim (x, s)$ implies by the Fishburn Rubinstein condition that $(w, t) \sim (x, t+s-r)$, and since $t+s-r$ is equivalent to v, we obtain $(w, t) \sim (x, v)$ as was to be shown.

For the next axiom, we introduce the following notation. For any r, s, t such that $r < s < t$, if for all y and z, $(y, r) \sim (z, s)$ holds if and only if $(y, s) \sim (z, t)$ holds, we write $r|s|t$. The notation emphasizes that the time period s is "midway" between r and t: the effect of the passage of time from r to s when comparing (y, r) and (z, s) is equivalent to the effect of the passage of time from s to t when comparing (y, s) and (z, t).

(A4) Weak Time-Transitivity. The preference relation $\succcurlyeq$ is transitive when restricted to all r, s and t such that $r|s|t$.

Just to clarify, the Weak Time-Transitivity axiom specifies that for all r, s and t such that $r|s|t$, we have that $(x, t_1) \succcurlyeq (y, t_2) \succcurlyeq (z, t_3)$ implies $(x, t_1) \succcurlyeq (z, t_3)$ for all $x, y, z \in X$ and $t_1, t_2, t_3 \in \{r, s, t\}$. Weak Time-Transitivity is a weaker property than transitivity, since it restricts preferences to be transitive only when considering three special periods. Continuing with the interpretation that once the individual knows how to compare choices between s and t when he knows how to compare choices in r and s, Weak Time-Transitivity is a natural property. Masatlioglu and Ok (2008) postulate that preferences are transitive whenever choices concern bundles in two periods. Thus, my transitivity requirement is stronger than theirs.

3. The two theorems

I now present my main result concerning dated bundles.

Theorem 1. *Suppose X is a compact, connected and separable space. The binary relation $\succsim$ is a time preference that satisfies Time Discounting, Weak Time-Transitivity, Orthogonality and Irrelevance iff there exist two continuous functions $u : X \to \mathbf{R}$ and $f : T^2 \to \mathbf{R}$ such that*

$$(x, t) \succcurlyeq (y, s), \quad iff\ u(x) \geq u(y) + f(t, s) \tag{1}$$

for all $(x, t), (y, s) \in X \times T$, and $f(\cdot, t)$ is increasing; $f(s, t) + f(t, s) = 0$; $2f(r, s) = f(r, t)$ whenever $f(r, s) = f(s, t)$.

The form of the representation in Eq. (1) is the same as in Masatlioglu and Ok (2008). More concretely, their prize space X is an open interval in $\mathbf{R}$, and their preference relations satisfy their set of axioms if and only if there exists an increasing homeomorphism $U : X \to \mathbf{R}_{++}$ and a continuous map $\eta : [0, \infty)^2 \to \mathbf{R}_{++}$ such that $(x, t) \succcurlyeq (y, s)$ iff $U(x) \geq U(y)\eta(t, s)$. Of course, by taking logarithms one can transform this expression into Eq. (1), and one can similarly transform representation (1) into that of Masatlioglu and Ok by exponentiating both sides of the inequality and letting $U = e^u$ and $\eta = e^f$.

The main differences between the representation of Masatlioglu and Ok and mine lies in the conditions that must be satisfied by the functions involved. Their theorem has two requirements on U and η which ensure that:

(i) a long enough passage of time compensates for any "increase" in the quality of the bundle, since $\eta(\infty, s) = 0$ for all s; that is, any bundle x at $t = \infty$ is worse than any date bundle (y, s);
(ii) there always exists bundles good enough (or bad enough) that will compensate for any passage of time, since U is a homeomorphism between X and $\mathbf{R}_{++}$; for any dated bundle (y, s) and time t, there exists x and x' such that $(x, t) \succ (y, s) \succ (x', t)$.

On the other hand, I have the requirement that $2f(r, s) = f(r, t)$ whenever $f(r, s) = f(s, t)$. While this requirement will be satisfied by all transitive and Weak Time transitive preferences, it may pose a problem in very specific instances of subadditive discounting. This restriction in the functional form of f reflects the fact that if the passage of time from r to s has the same effect as the passage of time from s to t, then the passage of time from r to t "should" be twice the effect of the passage from r to s.

Theorem 1 is important for two reasons. It provides a representation similar to that of Masatlioglu and Ok (2008) for the context of compact, connected separable spaces, which were not covered in their work. But most importantly, it allows the extension to the setup of choices over lotteries. This is relevant because the work of Masatlioglu and Ok is not suitable for that setup. All the applications and examples of Masatlioglu and Ok are also applications and examples of this Theorem 1, so the interested reader is referred to their work.

3.1. Risk

We now turn to the representation of preferences over dated lotteries. Formally, let Z be a compact, connected, separable, metric space. We will let X denote the space of probability measures over Z, endowed with the topology of weak convergence.[2] For the representation of preferences over lotteries, we add the standard independence axiom.

Independence. The preference relation $\succcurlyeq_0$ satisfies the standard independence axiom: for all $x, y, z \in X$ and $\alpha \in [0, 1]$, $x \succcurlyeq_0 y$ implies $\alpha x + (1-\alpha)z \succcurlyeq_0 \alpha y + (1-\alpha)z$.

Although Independence has been criticized from the descriptive viewpoint, Expected Utility is still the single most used tool in choice with risky alternatives. Therefore, using the Independence axiom is thus a natural starting point in the development of the theory of time preferences over risky alternatives. Moreover, the proof of the following theorem, my main result for risky choice, shows that there is no interaction between Independence and the other axioms. This implies that the present theory can be extended without difficulty to encompass other standard models of risky choice. In the following Theorem, for a lottery x and a function v, $E_x v$ denotes the expected value of v, according to lottery x.

Theorem 2. *The binary relation $\succsim$ is a time preference that satisfies Time Discounting, Weak Time-Transitivity, Orthogonality, Irrelevance and Independence iff there exist three continuous functions $u : [0, 1] \to \mathbf{R}$, $v : Z \to [0, 1]$ and $f : T^2 \to \mathbf{R}$ such that*

$$(x, t) \succcurlyeq (y, s) \quad iff \quad u(E_x v) \geq u(E_y v) + f(t, s)$$

for all $(x, t), (y, s) \in X \times T$, and $f(\cdot, t)$ is increasing; $f(s, t) + f(t, s) = 0$; $2f(r, s) = f(r, t)$ whenever $f(r, s) = f(s, t)$.

As a simple application of Theorem 2, one can replicate the results concerning Rubinstein Bargaining in Masatlioglu and Ok, where the set A of all agreements to the bargaining problem is the set of all probability measures λ on [0,1] with finite support.[3] In this case, if the agreement λ is reached, then with probability $\lambda(\{a\})$ player 1 will receive a, and player 2 receives $1 - a$.

4. Proofs

4.1. Proof of Theorem 1

The basic idea of the proof is to define $f(0, \cdot)$, and then use Irrelevance and $f(0, \cdot)$ to build $f(t, \cdot)$ for all other t. Before turning to the proof, let me first prove a lemma, about a new axiom, that will be useful in the sequel.

Strong Irrelevance (SI). For all $w, x, y, z \in X$ and $r, s, t, v \in T$, $r < s, t < v$:

$$\begin{array}{lcl} (y, r) \sim (z, s) & & (w, r) \succcurlyeq (x, s) \Leftrightarrow (w, t) \succcurlyeq (x, v) \\ \text{and} & \Rightarrow & \text{and} \\ (y, t) \sim (z, v) & & \text{for all } p \in [r, s] \text{ there is } q \in [t, v] \text{ such that } (w, r) \succcurlyeq (x, p) \Leftrightarrow (w, t) \succcurlyeq (x, q) \end{array}$$

Lemma 0. *Suppose X is a compact, connected and separable space and that the binary relation $\succsim$ is a time preference that satisfies Irrelevance and Time Discounting. Then it satisfies Strong Irrelevance.*

Proof. Fix any $p \in [r, s]$ and suppose that $(y, r) \sim (z, s)$ and $(y, t) \sim (z, v)$ are satisfied. By continuity $U = \{(w, p) : (w, p) \succcurlyeq (y, r)\}$ is closed, and by Time Discounting, $(z, p) \in U$. Similarly, $(y, p) \in L = \{(w, p) : (y, r) \succcurlyeq (w, p)\}$, which is is closed. Since X is connected, there exists $(z', p) \in U \cap L$, so that $(y, r) \sim (z', p)$. Using again continuity and Time Discounting, one can find $q \in [t, v]$ such that $(y, t) \sim (z', q)$. Then, Irrelevance, $(y, r) \sim (z', p)$ and $(y, t) \sim (z', q)$ imply that for all w and x,

$$(w, r) \succcurlyeq (x, p) \Leftrightarrow (w, t) \succcurlyeq (x, q)$$

as was to be shown. $\square$

[2] Recall that, under this topology, a sequence of probability measures (p_n) converges to p if and only if $\int_X f\, \mathrm{d}p_n \to \int_X f\, \mathrm{d}p$ for all continuous real valued functions in Z.

[3] I thank a referee for suggesting this application. The calculations are omitted because they are almost identical to those in Masatlioglu and Ok.

4.1.1. Defining $f(0,\cdot)$

If for all $t\in T$, $(x,0)\sim(x,t)$ the problem is trivial, so assume that for some $\bar{t}$, $(x,0)\succ(x,\bar{t})$. Likewise, if for all x, $y(x,0)\sim(y,0)$ the problem is trivial, so assume that for some a $\succcurlyeq_0-$ maximal $\bar{x}$ and minimal $\underline{x}$, $(\bar{x},0)\succ(\underline{x},0)$.

Step 1. Let $t_0=0$ and fix any t_1 such that for some x, $(x,0)\succ(x,t_1)$ and $(\bar{x},t_1)\succcurlyeq(\underline{x},0)$. One can easily show that since X is connected and separable, and $\succcurlyeq$ is continuous, $X\times\{t_0,t_1\}$ is $\succcurlyeq-$ separable (there is a countable set $Z\subset X$ such that whenever $(x,t)\succ(y,s)$ for $t,s\in\{t_0,t_1\}$ there is a $z\in Z$ such that $(x,t)\succcurlyeq(z,r)\succcurlyeq(y,s)$ for $r,t,s\in\{t_0,t_1\}$) and that therefore, there exists a continuous $U:X\times\{t_0,t_1\}$ such that $(x,t)\succcurlyeq(y,s)$ iff $U(x,t)\geq U(y,s)$ for all x, y in X and $s,t\in\{t_0,t_1\}$. The U function has the property that $U(x,t_0)>U(x,t_1)$ for all x, so that if we let $u\equiv U(x,t_0)$ and $h(u)\equiv U(x,t_1)$, by the solution to the Abel functional equation given in Kuczma's Theorem, (see Kuczma et al., 1990) we know that there exists a function g such that $g(u)=g(h(u))+$ constant. This means that there exists a $u_1:X\to\mathbf{R}$ such that $(x,t)\succcurlyeq(y,s)$ iff $u_1(x)\geq u_1(y)+f(t,s)$. Multiplying both sides by a constant one can normalize $f(t_0,t_1)=-f(t_1,t_0)=-1$ and by subtracting a constant to u_1 one can normalize $u_1(\bar{x})=1$.

Step 2. We start this step with two Lemmas.

Lemma 2. *Suppose X is a compact, connected and separable space and that the binary relation $\succsim$ is a time preference. Fix any $r,t\in T$ such that $r<t$* and $(\bar{x},t)\succcurlyeq(\underline{x},r)$. *Then, there exists s such that $r|s|t$.*

Proof. If t is equivalent to r, in the sense that $(x,t)\sim(x,r)$ for all x, there is nothing to prove, since all $s\in(r,t)$ are such that $r|s|t$. Therefore, suppose that $(x,r)\succ(x,t)$ for all x. Then, by continuity and connectedness of X and $(\bar{x},t)\succcurlyeq(\underline{x},r)\succ(\underline{x},t)$ there exists $\tilde{x}$ such that $(\underline{x},r)\sim(\tilde{x},t)$. Let $[\underline{x},\tilde{x}]\equiv\{x:\tilde{x}\succcurlyeq_0 x\succcurlyeq\underline{x}\}$. Define the following two correspondences from $[\underline{x},\tilde{x}]$ to $[r,t]$:

$$G_1(x)=\{s:(\underline{x},r)\sim(x,s)\}\quad\text{and}\quad G_2(x)=\{s:(\underline{x},s)\sim(x,t)\}.$$

That G_1 and G_2 are nonempty valued (i.e. well defined correspondences) follows from continuity and connectedness of X. We now show that G_1 has a closed graph (the proof for G_2 is analogous and omitted). Take any net $(x_\alpha,s_\alpha)\to(x,s)$ in the graph of G, that is, $(\underline{x},r)\sim(x_\alpha,s_\alpha)$ for all α. Using continuity we obtain $(\underline{x},r)\sim(x,s)$ so that the graph is closed.

Define $G=G_1-G_2$, and recall that the lower inverse of G is defined by $G^l(A)=\{x:G(x)\cap A\neq\emptyset\}$. Since $r\in G_1(\underline{x})$ and $t\in G_2(\underline{x})$, $r-t\in G(\underline{x})$, so that $G^l([r-t,0])$ is nonempty. We also know that since G has a closed graph, $G^l([r-t,0])$ is closed.[4] Similarly, $G^l([0,t-r])$ is nonempty and closed, so that connectedness of X ensures that there exists x^* such that $0\in G(x^*)$, which means that for some s, $(\underline{x},r)\sim(x^*,s)$ and $(\underline{x},s)\sim(x^*,t)$ as was to be shown. □

Lemma 3. *Suppose X is a compact, connected and separable space and that the binary relation $\succsim$ is a time preference that satisfies Time Discounting. For any $x,y\in X$ and $r,t\in T$ such that $(x,r)\succcurlyeq(y,t)$. Then, there exists w such that for $r|s|t$, $(x,r)\succcurlyeq(w,s)\succcurlyeq(y,t)$.*

Proof. If $(x,r)\succcurlyeq(y,s)\succcurlyeq(y,t)$, we are done and there is nothing to prove, so suppose $(y,s)\succ(x,r)\succcurlyeq(y,t)$. Since by Time Discounting we also have $(x,r)\succcurlyeq(x,s)$ we obtain $(y,s)\succ(x,r)\succcurlyeq(x,s)$. Since the sets $U=\{z:(z,s)\succcurlyeq(x,r)\}$ and $L=\{z:(x,r)\succcurlyeq(z,s)\}$ are nonempty and closed and X is connected, there must exist w such that $(x,r)\sim(w,s)$ as was to be shown. □

By Lemma 1, there exists $t_{1/2}$ such that $t_0|t_{1/2}|t_1$. Putting $r=t_0$, $s=t_{1/2}$ and $w=t_1$ in Strong Irrelevance, we note that

$$(x,t_0)\succcurlyeq(y,t_{1/2})\Leftrightarrow(x,t_{1/2})\succcurlyeq(y,t_1).$$

[4] By the Closed Graph Theorem G is upper hemicontinuous, and by Lemma 16.4 of Aliprantis and Border (1999), the lower inverse of closed sets is closed.

Therefore, as in Step 1 we obtain a function u_2 such that

$$(x, t_0) \succcurlyeq (y, t_{1/2}) \Leftrightarrow u_2(x) \geq u_2(y) + f(t_0, t_{1/2}) \quad (x, t_{1/2}) \succcurlyeq (y, t_1) \Leftrightarrow u_2(x) \geq u_2(y) + f(t_0, t_{1/2})$$

with $f(t_0, t_{1/2}) = -f(t_{1/2}, t_0) = -1/2$.

Moreover, u_2 also represents preferences over t_0, t_1. Fix $(x, t_0) \succcurlyeq (y, t_1)$ and according to Lemma 2 pick any w such that

$$(x, t_0) \succcurlyeq (w, t_{1/2}) \succcurlyeq (y, t_1).$$

Then, we obtain

$$\left.\begin{array}{r} u_2(x) \geq u(w) + f(t_0, t_{1/2}) \\ u_2(w) \geq u_2(y) + f(t_0, t_{1/2}) \end{array}\right\} \Rightarrow u_2(x) \geq u_2(y) + 2f(t_0, t_{1/2}).$$

The converse (from utility to preference) is treated similarly. Also, we normalize $u_2(\bar{x}) = 1$.

Step 3. Find $t_{1/4}$ such that $t_0|t_{1/4}|t_{1/2}$ and $t_{3/4}$ such that $(x, t_0) \succcurlyeq (y, t_{1/4}) \Leftrightarrow (x, t_{1/2}) \succcurlyeq (y, t_{3/4})$. By Lemma 0 and Strong Irrelevance, such a $t_{3/4}$ exists. Find u_3 such that

$$(x, t_0) \succcurlyeq (y, t_{1/4}) \Leftrightarrow u_3(x) \geq u_3(y) + f(t_0, t_{1/4})$$

for $f(t_0, t_{1/4}) = -f(t_{1/4}, t_0) = -1/4$. Note that, as in Step 2,

$$(x, t_0) \succcurlyeq (y, t_{1/2}) \Leftrightarrow u_3(x) \geq u_3(y) + 2f(t_0, t_{1/4}).$$

Moreover, if $(x, t_0) \succcurlyeq (y, t_{3/4})$, we can find w such that

$$(x, t_0) \succcurlyeq (w, t_{1/2}) \succcurlyeq (y, t_{3/4})$$

and hence,

$$\left.\begin{array}{r} u_3(x) \geq u_3(w) + 2f(t_0, t_{1/4}) \\ u_3(w) \geq u_3(y) + f(t_0, t_{1/4}) \end{array}\right\} \Rightarrow u_3(x) \geq u_3(y) + 3f(t_0, t_{1/4}).$$

Finally, it is also easy to show that .

$$(x, t_0) \succcurlyeq (y, t_1) \Leftrightarrow u_3(x) \geq u_3(y) + 4f(t_0, t_{1/4}).$$

We normalize $u_3(\bar{x}) = 1$.

Step n. One can continue in this fashion, and in Step n, find $t_{1/2^{n-1}}$ such that $t_0|t_{1/2^{n-1}}|t_{1/2^{n-2}}$ and for all k, set $t_{2k-1/2^{n-1}}$ such that

$$(x, t_0) \succcurlyeq (y, t_{1/2^{n-1}}) \Leftrightarrow (x, t_{2k/2^{n-1}}) \succcurlyeq (y, t_{(2k-1)/2^{n-1}}).$$

Also, one can find u_n such that for $k \leq 2^{n-1}$, $(x, t_0) \succcurlyeq (y, t_{k/2^{n-1}})$ if and only if

$$u_n(x) \geq u_n(y) + kf(t_0, t_{1/2^{n-1}}) = u_n(y) - k/2^{n-1}.$$

It is important to notice that the sequence t_{k_n} is increasing in k_n. Also, we normalize $u_n(\bar{x}) = 1$.

Taking Limits. In the next claim we will show that $u_n \to u$ for some continuous $u : X \to \mathbf{R}$. We will now show that u represents preferences on $X \times \{0, t\}$ for all $t \leq t_1$. Let

$$\tau = \left\{ t : \exists k, n \text{ such that } t = t_{k/n} \right\}.$$

Take then any $t \leq t_1$ and suppose $(x, t_0) \succcurlyeq (y, t)$. We have that for all $t_{k_n} \in \tau$, with $t_{k_n} \geq t$, $(x, t_0) \succcurlyeq (y, t_{k_n})$. Moreover, for some k,

$$\lim_{k_n \searrow k} t_{k_n} = t$$

so that for all n

$$(x, t_0) \succcurlyeq (y, t_{k_n}) \Leftrightarrow u_n(x) \geq u_n(y) - k_n \Rightarrow u(x) \geq u(y) - k.$$

That is, u represents preferences on $0, t$ for all $t \leq t_1$. With the procedure just described, one defines $f(0, t)$ for all $t \leq t_1$.

Showing that $u_n \to u$. Endow the space of continuous bounded functions on X with the sup norm. This is a complete metric space. We will show that the sequence $\{u_n\}$ is Cauchy, so that there exists a continuous u such that $u_n \to u$. Fix any $\varepsilon > 0$. If $t_{1/2^{N-1}}$ is small enough (N large enough), for all x in X there exists y such that

$$(x, 0) \sim (y, t_{1/2^{N-1}}) \quad \text{or} \quad (x, t_{1/2^{N-1}}) \sim (y, 0).$$

Take $M > \max\{N, 1 - (\log \varepsilon / \log 2)\}$, and fix $m \geq n \geq M$. Since u_n and u_m represent $\succcurlyeq_0$ we have that $u_m = g \circ u_n$ for some increasing g.

Lemma 4. *Let $K \in \mathbf{N}$ be the maximal k such that $1 - (K/2^{n-1})$ is in the range of u_n and of $u_n - (1/2^{n-1})$. Then, for all $k = 0, 1, \ldots, K$, k is in the range of u_n and of $u_n - (1/2^{n-1})$ and*

$$g\left(1 - \frac{k}{2^{n-1}}\right) = 1 - \frac{k}{2^{n-1}} \tag{2}$$

Proof. We first show that K is well defined, and $K \geq 1$. Since we have chosen a large enough n, there exists x_1 such that $(x_1, 0) \sim (\bar{x}, t_{1/2^{n-1}})$, so that

$$u_n(x_1) = u_n(\bar{x}) - \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}.$$

Since $u_n(\bar{x}) = 1$, we obtain that $1 - (1/2^{n-1})$ is in the range of u_n and of $u_n - (1/2^{n-1})$ establishing that $K \geq 1$.

We now show by induction that for all $k \leq K$, k is in the range of u_n and of $u_n - (1/2^{n-1})$ and that equation (2) is satisfied.

For $k = 0$, Eq. (2) is satisfied since we have normalized $u_n(\bar{x}) = u_m(\bar{x}) = g(u_n(\bar{x})) = 1$, and therefore we have $g_1(1) = 1$. For $k = 1$, Eq. (2) is satisfied because

$$u_n(x_1) = u_n(\bar{x}) + f(0, t_{1/2^{n-1}}) = u_n(\bar{x}) - \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}} \tag{3}$$

and

$$u_m(x_1) = u_m(\bar{x}) - \frac{1}{2^{n-1}} \Leftrightarrow g(u_n(x_1)) = g(u_n(\bar{x})) - \frac{1}{2^{n-1}} = g(1) - \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}. \tag{4}$$

Therefore, using Eqs. (3) and (4) we obtain

$$g\left(1 - \frac{1}{2^{n-1}}\right) = 1 - \frac{1}{2^{n-1}}.$$

Suppose now that we have shown that for some $k - 1 < K$, $1 - (k - 1/2^{n-1})$ is in the range of u_n and of $u_n - (1/2^{n-1})$ and that

$$g\left(1 - \frac{k-1}{2^{n-1}}\right) = 1 - \frac{k-1}{2^{n-1}}.$$

We now show that $1 - (k/2^{n-1})$ is in the range of u_n and of $u_n - (1/2^{n-1})$ and that

$$g\left(1 - \frac{k}{2^{n-1}}\right) = 1 - \frac{k}{2^{n-1}}.$$

Since 1 and $1 - (K/2^{n-1})$ are in the range of u_n and u_n is continuous, $1 - (k/2^{n-1})$ is also in the range of u_n. Similarly, $1 - (1/2^{n-1})$ and $1 - (K/2^{n-1})$ are in the range of $u_n - (1/2^{n-1})$ and since $u_n - (1/2^{n-1})$ is continuous, $1 - (k/2^{n-1})$

is also in the range of $u_n - (1/2^{n-1})$. Then, there exist x and y such that

$$u_n(x) = u_n(y) - \frac{1}{2^{n-1}} = 1 - \frac{k}{2^{n-1}} \Rightarrow u_n(y) = 1 - \frac{k-1}{2^{n-1}}.$$

We obtain

$$\begin{aligned} u_m(x) = u_m(y) - \frac{1}{2^{n-1}} &\Leftrightarrow g(u_n(x)) = g(u_n(y)) - \frac{1}{2^{n-1}} \Leftrightarrow g\left(u_n(y) - \frac{1}{2^{n-1}}\right) \\ &= g\left(u_n(y)\right) - \frac{1}{2^{n-1}} \Leftrightarrow g\left(1 - \frac{k-1}{2^{n-1}} - \frac{1}{2^{n-1}}\right) = g\left(1 - \frac{k-1}{2^{n-1}}\right) - \frac{1}{2^{n-1}} \Leftrightarrow g\left(1 - \frac{k}{2^{n-1}}\right) \\ &= g\left(1 - \frac{k-1}{2^{n-1}}\right) - \frac{1}{2^{n-1}} = 1 - \frac{k-1}{2^{n-1}} - \frac{1}{2^{n-1}} = 1 - \frac{k}{2^{n-1}} \end{aligned}$$

as was to be shown. □

Take now any $x \in X$. We will show that $|u_m(x) - u_n(x)| < \varepsilon$, proving that $\{u_n\}$ is Cauchy. Two cases must be considered.

Case 1. Not too low x. $1 - (k/2^{n-1}) \leq u_n(x) \leq 1 - (k - 1/2^{n-1})$ for some $k \leq K$. Let y, z be such that $u_n(y) = u_n(z) - (1/2^{n-1}) = 1 - (k/2^{n-1})$. By Lemma 3, we know that

$$u_m(y) = g(u_n(y)) = g\left(1 - \frac{k}{2^{n-1}}\right) = 1 - \frac{k}{2^{n-1}} = u_n(y)$$

Also,

$$u_m(z) = g(u_n(z)) = g\left(1 - \frac{k-1}{2^{n-1}}\right) = 1 - \frac{k-1}{2^{n-1}} = u_n(z).$$

Therefore, we obtain

$$u_n(y) \leq u_n(x) \leq u_n(z) \Leftrightarrow u_m(y) \leq u_m(x) \leq u_m(z) \Leftrightarrow 1 - \frac{k}{2^{n-1}} \leq u_m(x) \leq 1 - \frac{k-1}{2^{n-1}}$$

which implies

$$|u_m(x) - u_n(x)| \leq \frac{1}{2^{n-1}} < \varepsilon$$

as was to be shown.

Case 2. Low x. $u_n(x) < 1 - (K/2^{n-1})$. Let y be such that $u_n(y) = 1 - (K/2^{n-1})$ (such a y exists by definition of K). By Lemma 3, we have

$$u_m(y) = g(u_n(y)) = g\left(1 - \frac{K}{2^{n-1}}\right) = 1 - \frac{K}{2^{n-1}}.$$

We now show that $u_m(x) \geq u_m(y) - (1/2^{n-1})$. If $u_m(x) < u_m(y) - (1/2^{n-1})$, we would have that since u_m represents preferences on $X \times \{0, t_{1/2^{n-1}}\}$, $(y, t_{1/2^{n-1}}) \succ (x, 0)$, and therefore,

$$u_n(y) - \frac{1}{2^{n-1}} > u_n(x) \Leftrightarrow 1 - \frac{K+1}{2^{n-1}} > u_n(x)$$

which would contradict the maximality of K. Then, x satisfies

$$1 - \frac{K}{2^{n-1}} = u_m(y) > u_m(x) \geq u_m(y) - \frac{1}{2^{n-1}} = 1 - \frac{K+1}{2^{n-1}},$$

which together with $1 - (K+1)/(2^{n-1}) < u_n(x) < 1 - (K/2^{n-1})$ (maximality of K) implies

$$|u_m(x) - u_n(x)| \leq \frac{1}{2^{n-1}} < \varepsilon$$

as was to be shown.

Extending $f(0, t)$ **for** $t > t_1$. We will consider two types of $t > t_1$. We will consider first a $t^* > t_1$ such that $(\bar{x}, t^*) \succcurlyeq (\underline{x}, 0)$ and then the case in which t^* is such that $(\underline{x}, 0) \succ (\bar{x}, t^*)$.

For any $t^* > t_1$ such that $(\bar{x}, t^*) \succcurlyeq (\underline{x}, 0)$ we define, as in Steps $1 - n$ a sequence of times s_n such that $0|s_1|t^*$ and inductively $0|s_n|s_{n-1}$. This defines an infinite sequence of times s_n, since

- whenever t is not equivalent to 0, there exists $s < t$ such that $0|s|t$;
- $0|s|t$ implies that s is not equivalent to 0;
- t^* is not equivalent to 0, since t_1 is not equivalent to 0.

Lemma 5. *The sequence* $\{s_n\}$ *converges to some* s_L *which is equivalent to* 0.

Proof. Since the $\{s_n\}$ sequence is decreasing, it must have a limit, so suppose that contrary to the statement of the lemma, $s_n \to s_L$ with s_L not equivalent to 0. Let x and y be such that

$$(x, 0) \sim (y, s_1) \quad \text{and} (x, s_1) \sim (y, t), \tag{5}$$

which exist by definition of s_0. Let x_n and y_n be such that $(x_n, 0) \sim (y_n, s_{n+1})$ and $(x_n, s_{n+1}) \sim (y_n, s_n)$. Again, such x_n and y_n exist by definition of s_{n+1}. Since X is compact, we have that there exists convergent subsequences of x_n and y_n, $x_{n_k} \to x_L$ and $y_{n_k} \to y_L$ so that by continuity

$$(x_{n_k}, 0) \sim (y_{n_k}, s_{n_{k+1}}) \Rightarrow (x_L, 0) \sim (y_L, s_L).$$

Since s_L is not equivalent to 0, we must have $y_L \succ_t x_L$ for all time periods t.

We also have $(x_n, s_{n+1}) \sim (y_n, s_n)$ for all n, so that by continuity, $(x_L, s_L) \sim (y_L, s_L)$ which implies $y_L \sim_t x_L$ for all time periods t, contradicting our previous assertion that $y_L \succ_t x_L$. □

By Lemma 4 there exists an n such that $s_n \in [0, t_1]$, and we thus let $f(0, t^*) = 2^n f(0, s_n)$. From Steps1-n we know that u and $f(0, t^*)$ represent preferences on $X \times \{0, t^*\}$.

We now consider the case in which t^* is such that $(\underline{x}, 0) \succ (\bar{x}, t^*)$. Since preferences are continuous one can find a maximal $\bar{t}$ such that $(\underline{x}, 0) \sim (\bar{x}, \bar{t})$. Such a $\bar{t}$ has already been assigned an $f(0, \bar{t})$. We now let $f(0, t^*) = f(0, \bar{t}) - t^* + \bar{t}$. The structure of f beyond $\bar{t}$ doesn't really matter because all elements of X in 0 are strictly better than all elements of X in t^*.

4.1.2. Defining $f(t,s)$

One can use irrelevance to find an r such that $(x,0)\succcurlyeq(y,r)\Leftrightarrow(x,t)\succcurlyeq(y,s)$, so that $f(t,s)=f(0,r)$.

4.2. Proof of Theorem 2

By the von-Neumann Morgenstern Theorem (see for example the version in Dubra et al., 2004) there exists a $v:Z\to[0,1]$ such that $x\succcurlyeq_0 y$ if and only if $E_x v\geq E_y v$. Now we define a preference relation R on $[0,1]\times T$ by $(j,t)R(k,s)$ if and only if there exist x and y such that $(x,t)\succcurlyeq(y,s)$, $E_x v=j$ and $E_y v=k$. We will now show, in Lemma 5, that $\succcurlyeq$ is a time preference that satisfies TD, WTT, O and Irrelevance if and only if R is a time preference that satisfies TD, WTT, O and Irrelevance. Then, the proof will be complete since we then have that whenever $\succcurlyeq$ is a time preference that satisfies TD, WTT, O, Irrelevance and Independence,

$$(x,t)\succcurlyeq(y,s)\overset{\text{def. of }R}{\Leftrightarrow}(E_x v,t)R(E_y,s)\overset{L.5,Th.1}{\Leftrightarrow}u(E_x v)\geq u(E_y v)+f(t,s)$$

for all $(x,t),(y,s)\in X\times T$, and $f(\cdot,t)$ is increasing; $f(s,t)+f(t,s)=0$; $2f(r,s)=f(r,t)$ whenever $f(r,s)=f(s,t)$.

Lemma 6. *The binary relation $\succsim$ is a time preference on $X\times T$ that satisfies Time Discounting, Weak Time-Transitivity, Orthogonality and Irrelevance iff R is a time preference on $[0,1]\times T$ that satisfies Time Discounting, Weak Time-Transitivity, Orthogonality and Irrelevance.*

Proof of Lemma 5. Before proceeding with the proof of the main point of the Lemma, we note the following fact. for every $j,k\in[0,1]$ such that $(j,t)R(k,s)$, the x and y such that $(x,t)\succcurlyeq(y,s)$, $E_x v=j$ and $E_y v=k$ can be taken as linear combinations of a fixed $\succcurlyeq_0$- maximal $\bar{x}$ and $\succcurlyeq_0$-minimal $\underline{x}$. The standard construction of the utility function in the von-Neumann Morgenstern Theorem implies that if $E_x v=j$, then $x\sim_0 j\bar{x}+(1-j)\underline{x}$. Therefore, we assume without loss of generality that $x=j\bar{x}+(1-j)\underline{x}$.

Step 1. From $\succcurlyeq$ to R. Suppose $\succcurlyeq$ is a time preference that satisfies TD, WTT, O and I.

R is a time preference. Completeness. For any $(j,t),(k,s)\in[0,1]\times T$ there are $x,y\in X$ such that $E_x v=j$ and $E_y v=k$, and since $(x,t)\succcurlyeq(y,s)$ or $(y,s)\succcurlyeq(x,t)$ must hold, by definition of R, we obtain that $(j,t)R(k,s)$ or $(k,s)R(j,t)$ must hold.

Continuity. Fix $(j,t)\in[0,1]\times T$ and take any sequence $(k_n,s_n)\in[0,1]\times T$, with $(k_n,s_n)\to(k,s)$ and $(k_n,s_n)R(j,t)$ for all n. We then have that

$$\left.\begin{array}{l}(k_n\bar{x}+(1-k_n)\underline{x},s_n)\to(k\bar{x}+(1-k)\underline{x},s)\\ \text{and}\\ (k_n\bar{x}+(1-k_n)\underline{x},s_n)\succcurlyeq(j\bar{x}+(1-j)\underline{x},t)\end{array}\right\}\underset{\text{continuity of}\succcurlyeq}{\Rightarrow}(k\bar{x}+(1-k)\underline{x},s)\succcurlyeq(j\bar{x}+(1-j)\underline{x},t)$$

so that $(k,s)R(j,t)$. The case of the lower contour set is treated similarly.

Transitivity of R_0. Define R_0 as jR_0k iff $(j,0)R(k,0)$. Pick any $j,k,l\in[0,1]$ such that jR_0kR_0l. We then have

$$(j,0)R(k,0)R(l,0)\Leftrightarrow\left\{\begin{array}{l}\exists x,y,z\text{ with }E_x v=j,\ E_y v=k,\ E_z v=l\\ \text{and}\\ x\succcurlyeq_0 y\succcurlyeq_0 z\end{array}\right\}\underset{\text{transitivity of}\succcurlyeq_0}{\Rightarrow}(x,0)\succcurlyeq(z,0)\Rightarrow jR_0l$$

as was to be shown.

Equality of static preferences: $R_0 = R_1 = R_2 = \ldots$ We have

$$jR_0k \Leftrightarrow (j,0)R(k,0) \Leftrightarrow \left\{ \begin{array}{l} \exists x, y \text{ with } E_xv = j,\ E_yv = k \\ \text{and} \\ x \succcurlyeq_0 y \end{array} \Leftrightarrow \begin{array}{l} E_xv = j,\ E_yv = k \\ \text{and} \\ x \succcurlyeq_t y \end{array} \right\} \Leftrightarrow jR_tk$$

as was to be shown.

R satisfies Time Discounting. Define P to be the strict part of R and I its indifference: $(j,t)P(k,s)$ if and only if $(j,t)R(k,s)$ and not $(k,s)R(j,t)$; $(j,t)I(k,s)$ iff $(j,t)R(k,s)$ and $(k,s)R(j,t)$. Suppose $j,k,l \in [0,1]$ and $s,t,v \in T$, with $v \leq t$ and assume $(j,t)P(k,s)$. We have that for some x and y, $(x,t) \succ (y,s)$ and since $\succcurlyeq$ satisfies TD, $(x,v) \succ (y,s)$, so that $(j,v)P(k,s)$. The case of $(j,t)R(k,s)$ is treated similarly, establishing that R satisfies TD.

R satisfies Orthogonality. Take $j,k \in [0,1]$ and $s,t \in T$ and x,y such that $E_xv = j,\ E_yv = k$. Then since $\succcurlyeq$ satisfies Orthogonality,

$$(j,t)R(j,s) \Leftrightarrow (x,t) \succcurlyeq (x,s) \Leftrightarrow (y,t) \succcurlyeq (y,s) \Leftrightarrow (k,t)R(k,s).$$

R satisfies Irrelevance. For all $i, j, k, l \in [0,1]$ find $w, x, y, z \in X$ such that $E_wv = i$, $E_xv = j$, $E_yv = k$ and $E_zv = l$. Let $r, s, t, v \in T$, $r < s, t < v$:

$$\begin{array}{l} (k,r)I(l,s) \\ \text{and} \\ (k,t)I(l,v) \end{array} \Rightarrow \begin{array}{l} (y,r) \sim (z,s) \\ \text{and} \\ (y,t) \sim (z,v) \end{array} \Rightarrow \begin{array}{l} (w,r) \succcurlyeq (x,s) \Leftrightarrow (w,t) \succcurlyeq (x,v) \\ \text{and} \\ \forall p \in [r,s] \exists q \in [t,v] \text{ such that } (w,r) \succcurlyeq (x,p) \Leftrightarrow (w,t) \succcurlyeq (x,q) \end{array}$$

$$\Rightarrow \begin{array}{l} (i,r)R(j,s) \Leftrightarrow (i,t)R(j,v) \\ \text{and} \\ \forall p \in [r,s] \exists q \in [t,v] \text{ such that } (i,r)R(j,p) \Leftrightarrow (i,t)R(j,q) \end{array}$$

so that R satisfies Irrelevance.

Step 2. From R to $\succcurlyeq$. The proof is analogous and is omitted, so the proof of the Lemma is complete. □

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