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Journal of Economic Theory 144 (2009) 825–849

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

English auctions and the Stolper–Samuelson theorem ☆

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Received 6 February 2007; final version received 26 December 2007; accepted 8 July 2008

Available online 17 July 2008

Abstract

We prove that the English auction (with bidders that need not be ex ante identical and may have interdependent valuations) has an efficient ex post equilibrium. We establish this result for environments where it has not been previously obtained. We also prove two versions of the Stolper–Samuelson theorem, one for economies with n goods and n factors, and one for non-square economies. Similar assumptions and methods underlie these seemingly unrelated results.

JEL classification: C60; D44; F11

Keywords: Auction theory; International trade; Monotone comparative statics; Global univalence

1. Introduction

A similar mathematical structure, comparative statics of the solution of a system of equations, underlies diverse economic results such as the efficiency of the English auction and the Stolper–

☆ The results in this paper were circulated earlier as two separate papers: "Minimal Assumptions for Efficiency in Asymmetric English auctions," by Dubra, and "Comparative Statics, English auctions and the Stolper–Samuelson theorem," by Echenique and Manelli. We thank Rabah Amir, Sergei Izmalkov, Vijay Krishna, Preston McAfee, Peter Neary, John Quah, Kevin Reffett, and Chris Shannon for useful comments. Manelli's research was partially supported by NSF grants SES-0095524 and SES-0241373.
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0022-0531/$ – see front matter
doi:10.1016/j.jet.2008.07.001

Samuelson theorem. We find that related assumptions on the system of equations allow us to extend, significantly, the domain of application of both results. We prove that the English auction has an efficient ex post equilibrium in environments where this result had not been previously obtained. We also prove versions of the Stolper–Samuelson theorem for economies with more than two goods and factors, and for non-square economies.

Consider the system of equations

$$\begin{array}{lcl} v_1(s_1, s_2, \ldots, s_n) = & p_1, \\ v_2(s_1, s_2, \ldots, s_n) = & p_2, \\ \vdots \qquad\quad \vdots & \vdots \\ v_n(s_1, s_2, \ldots, s_n) = & p_n, \end{array} \tag{1}$$

where $s_1, s_2, \ldots, s_n$ are the unknowns, $p_1, p_2, \ldots, p_n$ are the parameters, and $v_1, \ldots, v_n$ are functions. How does the solution $(s_1, \ldots, s_n)$ respond to changes in the parameters $(p_1, \ldots, p_n)$? To provide a meaningful answer to this classic question, restrictions must be imposed on the functions $v_1, \ldots, v_n$.

Our assumptions have the flavor of a "relative sensitivity" requirement. Suppose each function v_i is relatively more sensitive to one variable (which we will call "its own" variable) than to the others. If the effect of such variable s_i on v_i is an "own" effect, we might require that the own effect be relatively more important than the "cross" effect, the effect of s_i on v_j. Suppose for instance that $n = 2$ and that $v_1(s_1, s_2)$ and $v_2(s_1, s_2)$ are increasing functions. Let parameters change so that $p_1' > p_1$ and $p_2' < p_2$. Since v_1 and v_2 are increasing, s_1 and s_2 cannot both increase or decrease simultaneously. Our "relative sensitivity" implies, as a consequence, that $s_1' > s_1$ and $s_2' < s_2$. We warn the reader that our formal assumption varies with the application considered, and that it differs from the discursive version in this paragraph. Still, the intuitive rendition illustrates its use.

We now describe how a similar formal structure underlies both applications, the efficiency of the English auction and the Stolper–Samuelson theorem. For expositional ease, we begin with the latter.

Consider an economy with two goods, two factors of production, and constant-returns-to-scale technologies. Let $v_i(s_1, s_2)$, $i = 1, 2$, be the per-unit cost of producing good i given factor prices (s_1, s_2). If output prices (p_1, p_2) are exogenously determined—the standard small-country assumption in international trade—an equilibrium in the factors' markets is the solution to

$$\begin{aligned} v_1(s_1, s_2) &= p_1, \\ v_2(s_1, s_2) &= p_2. \end{aligned} \tag{2}$$

The interpretation of (2) is that there are no extraordinary profits in the production of goods 1 and 2—a consequence of the combined assumptions, standard in international trade theory, that both goods are produced in equilibrium and that the technologies are constant returns to scale.

The Stolper–Samuelson theorem states that if the production of good 1 is relatively more intense in the use of factor 1, an exogenous increase in the price of good 1 brings about an increase in the price of factor 1 and a decrease in the price of factor 2. Let K and L represent factors 1 and 2 respectively, and let $K_i(s_1, s_2)$ and $L_i(s_1, s_2)$ be the cost-minimizing quantities of factors in sector i when factor prices are (s_1, s_2). Stolper and Samuelson's factor-intensity assumption is that

$$\frac{K_1(s_1, s_2)}{L_1(s_1, s_2)} > \frac{K_2(s_1, s_2)}{L_2(s_1, s_2)}. \tag{3}$$

Using Shepard's lemma, however, the factor-intensity assumption can be stated as

$$\frac{\partial v_1(s_1, s_2)/\partial s_1}{\partial v_1(s_1, s_2)/\partial s_2} > \frac{\partial v_2(s_1, s_2)/\partial s_1}{\partial v_2(s_1, s_2)/\partial s_2}$$

for all (s_1, s_2). It is this formulation in terms of average cost functions that facilitates our approach. Stolper and Samuelson's factor-intensity condition is an instance of our "relative sensitivity" property. Expressed as an inequality of factor ratios, the factor-intensity condition does not readily generalize to economies with more than two goods and two factors. We will make our "relative sensitivity" assumption on the average cost curves, and this allows us to extend the notion of factor-intensity to economies with more than two factors, and to non-square economies. In turn, this leads to new versions of the Stolper–Samuelson theorem.

We turn to our auctions application. Consider an English auction where two (ex ante) different bidders have interdependent valuations. Bidder i, $i = 1, 2$, only observes her private signal s_i before the auction, and i's valuation for the object is $v_i(s_1, s_2)$. If p is the price quoted by the auctioneer, a solution $(s_1(p), s_2(p))$ to the system

$$\begin{aligned} &v_1(s_1, s_2) = p, \\ &v_2(s_1, s_2) = p \end{aligned} \tag{4}$$

indicates that both bidders are indifferent between getting the object and abandoning the auction. If the solution $(s_1(p), s_2(p))$ to (4) is increasing in p, then it has an inverse $(p_1(s_1), p_2(s_2))$. This inverse function can be used to construct bidding strategies: bidder i with signal s_i will remain in the auction until the auctioneer arrives at price $p_i(s_i)$. Under certain "relative sensitivity" assumptions on the value functions, it can be shown that these bidding strategies implement an efficient ex post equilibrium in the English auction.

The question of whether English auctions have efficient ex post equilibria in the environments described, was first posed by Maskin [10]. He studied the two-bidder case and assumed a "single crossing condition," namely that

$$\forall (s_1, s_2), \quad \frac{\partial v_1}{\partial s_1}(s_1, s_2) \geqslant \frac{\partial v_2}{\partial s_1}(s_1, s_2). \tag{5}$$

This condition captures the notion that s_i is more important for v_i than for v_j, $j \neq i$ and it therefore belongs to the "relative sensitivity" family of assumptions. Maskin proved that his condition implies that a solution $(s_1(p), s_2(p))$ to (4) exists, that it is unique and increasing, and that the implicit bidding strategies implement an efficient ex post equilibrium.

Maskin's result, however, does not extend to auctions with more than two bidders. Krishna [9] describes a three-bidder example, satisfying Maskin's single-crossing property (applied pairwise), where the English auction does not have an efficient equilibrium.[1] Our "relative sensitivity" allows us to prove the existence of ex post efficient equilibria with arbitrarily many bidders.

The reader will have noticed that Stolper and Samuelson's factor-intensity condition and Maskin's single crossing property are very similar.

[1] Krishna attributes the idea of the example to Phil Reny. Krishna also introduces alternative conditions to Maskin's single crossing property that restore the existence of efficient equilibria for environments with the n bidders. We compare our results to Krishna's below.

2. The Stolper–Samuelson theorem

We prove two versions of the Stolper–Samuelson theorem. The first one applies to square economies with n goods and factors of production; the second applies to non-square economies.

Consider a standard international-trade model. There are n non-traded, non-produced factors used in the production of n traded, final goods. Factor endowments are owned by consumers who offer them inelastically. Inputs are not consumed. A small-country assumption implies that the prices $\mathbf{p} = (p_1, \dots, p_n)$ of the n consumption goods are exogenously given. The endogenous vector of factor prices is denoted by $\mathbf{s} = (s_1, s_2, \dots, s_n)$. The production technology exhibits constant returns to scale. The unit cost of producing good i given factor prices $(s_1, s_2, \dots, s_n)$ is $v_i(s_1, s_2, \dots, s_n)$; the cost of producing y_i units of good i is then $v_i(\mathbf{s})y_i$. An equilibrium in this model is characterized by the zero-profit conditions: A combination of prices $(s_1, s_2, \dots, s_n, p_1, p_2, \dots, p_n)$ is an *equilibrium* if $p_i = v_i(s_1, s_2, \dots, s_n)$ for all i.[2]

Stolper and Samuelson [14] further assume that there are only two goods and two factors, that the per-unit cost functions v_i are differentiable, and that the production of good 1 is relatively more intense in the use of factor 1 as discussed in the Introduction. The thesis of their theorem is that an exogenous increase in the price of good 1 brings about an increase in the price of factor 1 and a decrease in the price of factor 2.

Of their extra assumptions, we only need the "relative sensitivity" assumption, a notion of factor-intensity that can be applied to more general economies than those studied by Stolper and Samuelson.

Let

$$P(\mathbf{s}' - \mathbf{s}) = \{i = 1, \dots, n\colon s'_i - s_i > 0\}.$$

The set $P(\mathbf{s}' - \mathbf{s})$ identifies the coordinates that have increased, the *strictly positive* coordinates of $(\mathbf{s}' - \mathbf{s})$.

Definition 1. The functions $v_1, v_2, \dots, v_n$ satisfy the *dominant-effect property* if, for any $\mathbf{s}$ and $\mathbf{s}'$ with $P(\mathbf{s}' - \mathbf{s}) \neq \emptyset$,

$$\max_{i \in P(\mathbf{s}'-\mathbf{s})} v_i(\mathbf{s}') - v_i(\mathbf{s}) > \max_{j \notin P(\mathbf{s}'-\mathbf{s})} v_j(\mathbf{s}') - v_j(\mathbf{s}).$$

The dominant-effect property is a relative-factor-intensity assumption. In the two-factor–two-good case, it states that if the price of factor 1 increases and the price of factor 2 decreases, then the cost of good 1 must increase more than the cost of good 2 (or the cost of good 2 must decrease more than the cost of good 1). That is to say, the production of every good i must be relatively more intense in the use of the factor i. Indeed, with differentiable cost functions and $n = 2$, the Stolper–Samuelson factor-intensity assumption 3 implies the dominant-effect property.

The dominant-effect property generalizes the notion of relative factor-intensity to economies with more than two goods and with non-differentiable cost functions. Suppose several factor prices change simultaneously. The dominant-effect property requires that *one* of the goods whose "corresponding factor-price" has increased must have a larger cost-increase than the cost-increase of any good whose "corresponding factor-price" decreased.

[2] Implicit is the assumption—standard in trade-theory—that all goods are produced in equilibrium: p_i could be less than $v_i(s_1, s_2, \dots, s_n)$ if good i were not produced.

Theorem 1. *Let $v_1, v_2, \ldots, v_n$ be non-decreasing average cost functions that satisfy the dominant-effect property. Let the price-vectors $(\mathbf{s}, \mathbf{p})$ and $(\mathbf{s}', \mathbf{p}')$ be equilibria. If $p'_i > p_i$ for some good i, and $p'_h \leqslant p_h$ for all $h \neq i$, then $s'_i > s_i$.*

If, in addition, the functions $v_1, v_2, \ldots, v_n$ are strictly increasing, then $s'_h < s_h$ for at least one $h \neq i$.[3]

Proof. Since $(v_1, v_2, \ldots, v_n)$ are non-decreasing and $p'_i > p_i$, it cannot be the case that $\mathbf{s}' \leqslant \mathbf{s}$. Therefore $P(\mathbf{s}' - \mathbf{s})$ is non-empty.

We prove that i is in $P(\mathbf{s}' - \mathbf{s})$. Suppose that i is not in $P(\mathbf{s}' - \mathbf{s})$. Then, by the dominant-effect property, for some $h \in P(\mathbf{s}' - \mathbf{s})$, $p'_h - p_h > p'_i - p_i$. A contradiction, since $p'_i - p_i > 0$ and $p'_h - p_h \leqslant 0$ for all $h \neq i$. We conclude that i is in $P(\mathbf{s}' - \mathbf{s})$.

If the functions $v_1, v_2, \ldots, v_n$ are strictly increasing, then $\mathbf{s}' > \mathbf{s}$ implies $p'_h > p_h$ for all h. Since $p'_h \leqslant p_h$ for all $h \neq i$, it cannot be the case that $\mathbf{s}' > \mathbf{s}$. Thus, there is h such that $s'_h - s_h < 0$; h cannot be equal to i, as $s'_i - s_i > 0$. □

In the two-factor–two-good case, Theorem 1 states that if a country opens up to trade and as a consequence p_1 increases while p_2 either decreases or stays the same, then the price of factor 1 will increase and the price of factor 2 will decrease. Thus the owners of factor 1 will gain and the owners of factor 2 will lose from opening up to trade.

In the n-factor–n-good case, Theorem 1 states that, if p_1 increases, and p_h either decreases or stays the same, for all other goods h, then the owners of factor 1 will gain, and the owners of at least one of the other factors will lose. Note that the thesis of Theorem 1 is *weaker* in this case because it does not say that $s'_h < s_h$ for all $h \neq i$.[4]

Theorem 1 delivers the message of the Stolper–Samuelson theorem in considerable generality. It is global because, unlike the Stolper–Samuelson theorem, it applies to any changes in prices, not only infinitesimal changes. In summary, the differences between Theorem 1 and Stolper and Samuelson's statement are as follows.

(1) Stolper and Samuelson's relative factor-intensity condition for two goods is stronger than the dominant-effect property.
(2) Stolper and Samuelson's conclusion is local; the conclusion of Theorem 1 is global.
(3) Stolper and Samuelson require that the cost functions $(v_1, v_2, \ldots, v_n)$ be differentiable, and that the Implicit Function Theorem be applicable; Theorem 1 does not.
(4) Stolper and Samuelson's version of the theorem only holds when $n = 2$ (see, for example, Chipman [2]).

The original Stolper–Samuelson theorem, and its generalization in Theorem 1, share the assumption that the number of final goods is the same as the number of factors. This is probably unrealistic. We offer a generalization of the Stolper–Samuelson theorem that avoids this assumption. The generalization is obtained by simply varying the dimensionality of the variables s_i and p_j from scalars to vectors. Then, the same framework used in Theorem 1 yields the desired extension.

[3] A function $v_i(\mathbf{s})$ is *strictly increasing* if $\mathbf{s}' > \mathbf{s}$ implies $v_i(\mathbf{s}') > v_i(\mathbf{s})$; it is *non-decreasing* if $\mathbf{s}' \geqslant \mathbf{s}$ implies $v_i(\mathbf{s}') \geqslant v_i(\mathbf{s})$.

[4] Chipman [2] calls this statement the weak Stolper–Samuelson theorem.

We can identify the technologies behind each v_i with a *sector*, producing a collection of m_i final goods. Hence, $v_i(\mathbf{s})$ is a vector in $\mathbf{R}^{m_i}$, and so is the corresponding parameter p_i in system (1). The vector p_i in $\mathbf{R}^{m_i}$ is the vector of prices for the m_i final goods produced by sector i. The total number of final goods produced is the sum of goods produced by all sectors, i.e., $M = \sum_{i=1}^{n} m_i$.

For each sector i there is a group of k_i factors that are used more intensively in sector i. (We make this precise below in the formal definition.) In terms of the model, the variable s_i belongs to $\mathbf{R}^{k_i}$; s_i represents the vector of factor-prices corresponding to the k_i 'factors used more intensively in sector i.' The total number of factors in the economy is simply the sum of all groups of factors, i.e. $K = \sum_{i=1}^{n} k_i$.

Each sector i has a constant-returns technology that can be represented by an average-cost function $v_i : \mathbf{R}^K \to \mathbf{R}^{m_i}$.

We have thus redefined system (1) so that all variables are vectors. If we set $k_i = 1 = m_i$ for all $i = 1, \ldots, n$, we are back in the framework of Theorem 1, an n-sector economy with n final goods and n factors. If in addition $n = 2$, we have the classic Stolper–Samuelson environment.

We now adapt the dominant-effect property to the new environment, and to do so, we must look at the set $P(\mathbf{s}' - \mathbf{s})$ that figures conspicuously in its definition. Given $\mathbf{s}'$, $\mathbf{s}$, and an order $\succ$, let

$$P(\mathbf{s}' - \mathbf{s}) = \{ i = 1, \ldots, n \colon s_i' - s_i \succ 0 \}.$$

In our previous applications when s_i was a scalar, the order $\succ$ was the standard order on the real line; the meaning of $s_i' \succ s_i$ is simply $s_i' > s_i$. In the new environment s_i is a k_i-dimensional vector and therefore we have options in defining $s_i' \succ s_i$. We choose the following order

$$s_i' \succ s_i \quad \text{if and only if} \quad s_i' \nleqslant s_i$$

yielding

$$P(\mathbf{s}' - \mathbf{s}) = \{ i = 1, \ldots, n \colon s_i' - s_i \nleqslant 0 \}.$$

We will briefly comment on alternative definitions of $\succ$ after we state the theorem.

We are now ready to state the adapted dominant-effect property. The functions $v_1, v_2, \ldots, v_n$ satisfy the adapted dominant-effect property if for any $\mathbf{s}'$ and $\mathbf{s}$ with $P(\mathbf{s}' - \mathbf{s}) \neq \emptyset$,

$$\max_{\{\ell=1,\ldots,m_i \colon i \in P(\mathbf{s}'-\mathbf{s})\}} v_{\ell i}(\mathbf{s}') - v_{\ell i}(\mathbf{s}) > \max_{\{\ell=1,\ldots,m_j \colon j \notin P(\mathbf{s}'-\mathbf{s})\}} v_{\ell j}(\mathbf{s}') - v_{\ell j}(\mathbf{s}),$$

where $v_{\ell j}(\mathbf{s})$ is the ℓth component of the m_j-dimensional vector $v_j(\mathbf{s})$.

The adapted dominant-effect property is simply an expression of the factor-intensity assumption, as discussed for square economies.

As before, we say that a pair of prices $(\mathbf{s}, \mathbf{p})$ is an equilibrium if $p_i = v_i(\mathbf{s})$ for $i = 1, \ldots, n$.

Theorem 2. *Let $v_1, v_2, \ldots, v_n$ be non-decreasing and satisfy the adapted dominant-effect property. Let $(\mathbf{s}, \mathbf{p})$ and $(\mathbf{s}', \mathbf{p}')$ be equilibria. If $p_i' > p_i$ for some good i, and $p_h' \leqslant p_h$ for all $h \neq i$, then there is $1 \leqslant \ell \leqslant k_i$ such that $s_{\ell i}' > s_{\ell i}$, where $s_{\ell i}$ is the price of factor ℓ within the group of factor prices s_i.*

The proof is similar to that of Theorem 1 and therefore we omit it. Alternative definitions of the order $\succ$ used, give rise to variations of the dominant-effect property and of the theorem above. For instance, if we strengthen the adapted dominant-effect property so that $s_i' \succ s_i$ if and only if $s_i' \gg s_i$, the theorem then concludes that $s_{\ell i}' > s_{\ell i}$ for all ℓ, $\ell = 1, \ldots, k_i$: the prices of all factors (in which sector i is intensive) increase.

We conclude the section with a discussion of related literature. There is a large literature on generalizations of the Stolper–Samuelson theorem. We refer the interested reader to Ethier [5] for a survey. The closest result to Theorem 1 is an application of the weak axiom of cost minimization (Ethier [5]). This application, however, barely retains the economic content of the Stolper–Samuelson theorem because it does not say which factor-prices change as a result of specific changes in goods prices.[5] In trade theory, predicting *who* will win (and thus favor) an opening to trade, is important. Contrary to Theorem 1, the application of the weak axiom only gives the standard "average correlation" result between goods and factor prices: on-average-higher good prices yield on-average-higher factor prices. On the other hand, the application of the weak axiom does not require assumptions on $\mathbf{v}$, it is purely a product of cost minimization.

When $n = 2$, Samuelson [13] also proved the Factor-Price Equalization Theorem: if $\mathbf{v}$ satisfies the relative factor-intensity condition, $\mathbf{v}(\mathbf{s})$ has a global inverse, so factor prices are uniquely determined by $\mathbf{p}$. In the context of trade, this implies that all countries that share the same technology must have the same factor prices. This is, arguably, an empirically less relevant proposition than the Stolper–Samuelson theorem, or than Theorem 1. When $n > 2$, the relative factor-intensity condition is not sufficient for the existence of a global inverse. Gale and Nikaido [6] proved that, if $\mathbf{v}$ is C^1, and the Jacobian of $\mathbf{v}$ is everywhere a P-matrix—all the principal minors of $\mathbf{v}$ are positive—then $\mathbf{v}$ is globally invertible. But even if the Jacobian is everywhere a P-matrix, the Stolper–Samuelson theorem need not hold (Chipman [2]). Theorem 1 shows that our generalization of the factor-intensity condition suffices to give the Stolper–Samuelson result with $n > 2$. We do not need to address the problem of the existence of a global inverse.

3. English auction

We study the irrevocable exit English auction introduced by Milgrom and Weber [12]. In this game the auctioneer continuously raises the asking price, starting from zero. A bidder has the option of quitting the auction publicly at any time. Once a bidder quits, the bidder cannot reenter. The last bidder to remain active is the winner and pays the price called at the time that the previous to the last bidder leaves the auction.

We prove that the English auction has an efficient ex post equilibrium in models where bidders may have interdependent valuations and need not be ex ante identical.

3.1. A sufficient condition

Let $N = \{1, 2, \dots, n\}$ be the set of players. Each player i observes a signal $s_i \in [0, b]$. This signal is only known to player i. Signals are drawn according to some probability measure μ over $[0, b]^n$ that need not posses a density. The signals affect the values that players have for the objects. Player i's valuation is a continuous function $v_i : [0, b]^n \to \mathbf{R}$ that maps profiles of signals (one for each player) into real numbers, with $v_i(\mathbf{0}) = 0$ for all i, and that is strictly increasing in its own signal, so that for all i and all $\mathbf{s}_{-i}$ in $[0, b]^{n-1}$, $s_i' > s_i$ implies $v_i(s_i', \mathbf{s}_{-i}) > v_i(s_i, \mathbf{s}_{-i})$. For any $\mathbf{s}$, let

$$W(\mathbf{s}) = \bigl\{i \in N\colon\ v_i(\mathbf{s}) \geqslant v_k(\mathbf{s})\ \forall k \in N\bigr\}.$$

[5] The comparison with Jones and Scheinkman's [8] "every factor has some natural enemy" result is similar. Jones and Mitra's [7] version of Stolper–Samuelson involves a dominant diagonal condition, which shares the spirit of the dominant-effect property. But they also require additional strong assumptions: that the profile of factor shares take an identical (up to a permutation) geometric decay form for all sectors.

We refer to $W(\mathbf{s})$ as the set of "winners" at $\mathbf{s}$ and denote by $|W(\mathbf{s})|$ the cardinality of $W(\mathbf{s})$.

We now define two properties that are jointly sufficient for the existence of an efficient equilibrium.

Definition 2. The functions $v_1, \ldots, v_n$ are *increasing at ties* if for every $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$ and all $i \in W(\mathbf{s})$

$$\left.\begin{array}{l}\mathbf{s}' \geqslant \mathbf{s},\\ s'_j > s_j \text{ implies } j \in W(\mathbf{s})\end{array}\right\} \quad \Rightarrow \quad v_i\left(s'_i, \mathbf{s}'_{-i}\right) \geqslant v_i\left(s'_i, \mathbf{s}_{-i}\right).$$

The interpretation of the above property is as follows. Suppose $\mathbf{s}$ is a profile of signals for which at least two players have equal and highest valuations. Then, the property requires that if one of the winner's signal increases to s'_i, then the effect of the other player's signals, when they increase from $\mathbf{s}_{-i}$ to $\mathbf{s}'_{-i}$, does not hurt player i. Example 5.6 of Maskin [11] (taken from Esö and Maskin [4]) shows that even if some sort of single crossing property is satisfied, one still needs that player j's signal does not affect player i's valuation "very" negatively, if an efficient equilibrium is to exist (an equilibrium is efficient if it always allocates the object to one of the players with the highest valuation). Valuations in Maskin's example are not increasing at ties, and that is why he finds that no efficient equilibrium exists. Stronger versions of this property have been standard in the literature. The most common assumption of this kind is that v_i is increasing in s_i and weakly increasing in $\mathbf{s}_{-i}$. The only paper in this literature that has an assumption that is not stronger than increasing at ties is in Krishna [9]. The assumption in that paper is neither weaker nor stronger than increasing at ties: it states that when i's signal increases, the sum of all player's valuations increases.

Definition 3. The functions $v_1, \ldots, v_n$ satisfy the *own effect property* (*OEP*) if for every $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$,

$$\left.\begin{array}{l}\mathbf{s}' \geqslant \mathbf{s}\\ s'_j > s_j \text{ implies } j \in W(\mathbf{s})\end{array}\right\} \quad \Rightarrow \quad \max_{j:\, s'_j > s_j} v_j(\mathbf{s}') \geqslant \max_{k:\, s'_k = s_k} v_k(\mathbf{s}').$$

It states that the effect of an increase in some signals is larger for one of the players whose signal increased than for all the rest of the players. Notice that it is a form of single crossing: if there are only two players, j's valuation is equal to k's and j's signal increases, j's valuation is larger than k's. As will be shown later, it is the weakest form of "single crossing" that has been used in this branch of the literature.

In the auction we study, a *strategy* for a player is a function that determines a price at which to quit, for each realization of the private information, and each history of who left the auction at what price. Formally, a strategy for bidder i is a collection of functions, one for each set of (active) players A and each profile $\mathbf{p}^{N\setminus A}$ of prices at which bidders in $N\setminus A$ quit the auction, $\beta_i^A : [0,b] \times \mathbf{R}_+^{N\setminus A} \to \mathbf{R}_+$ where $i \in A$, $|A| > 1$ and $\beta_i^A(s_i, \mathbf{p}^{N\setminus A}) > \max\{p_j\colon\ j \in N\setminus A\}$. The value $\beta_i^A(s_i, \mathbf{p}^{N\setminus A})$ is the price at which bidder i will drop out if players in $N\setminus A$ dropped at prices $\mathbf{p}^{N\setminus A}$ and nobody quits before. As long as $p < \beta_i^A(s_i, \mathbf{p}^{N\setminus A})$ he stays in the auction; he drops out when $p = \beta_i^A(s_i, \mathbf{p}^{N\setminus A})$; in any history in which $p > \beta_i^A(s_i, \mathbf{p}^{N\setminus A})$ he drops out (this part of the strategy will never be used). A profile of strategies is an *ex-post equilibrium* if it remains an equilibrium even if all players know everybody else's signals.

Theorem 3. *If $v_1, \dots, v_n$ are increasing at ties and satisfy the own effect property, then the English auction has an efficient ex post equilibrium.*

3.2. Necessity

Theorem 3 shows that the own-effect property is sufficient for the existence of equilibrium. We now prove that, under a regularity condition on valuations, there is a sense in which the own-effect property is also necessary for the existence of efficient equilibria in undominated strategies.

So far we have said that an equilibrium is efficient if it allocates the object to one of the players with the highest valuation for all profiles of signals. This definition of efficiency is the most demanding if one is concerned with finding sufficient conditions for the existence of an efficient equilibrium. But one could also use another definition of efficiency which is more demanding for necessity, and less so for sufficiency. Let us say that if an equilibrium of the English auction (with valuations v and distribution of signals μ) assigns the object to the highest bidder with μ-probability 1 it is *μ-efficient*.

Suppose now that we want to prove a theorem like "If property P of the profile of valuations v is violated, then there is no μ-efficient equilibrium for any μ." There is no hope for such a theorem, because if we assume $v_i(\mathbf{0}) = 0$ for all i and set $\mu(\mathbf{0}) = 1$, then any strategy profile that has $\beta_i^N(0, \emptyset) = 0$ (all players quit at $p = 0$, when all players are active, if they have a signal of 0) is a μ-efficient equilibrium. Hence, we will show a theorem of the form "If property P of the profile of valuations v is violated, then there is a μ such that no μ-efficient equilibrium exists."

Definition 4. The functions $v_1, \dots, v_n$ are *regular* if each v_i is twice continuously differentiable and for all $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ the Jacobian matrix of partial derivatives of subsets of the winners is invertible, or more formally, for all $P \subset W(\mathbf{s})$,

$$D_P v(\mathbf{s}) = \left(\frac{\partial v_i(\mathbf{s})}{\partial s_j} \right)_{i,j \in P} \tag{6}$$

is invertible.

We now show that in the presence of the regularity assumption above, the own effect property is necessary.[6] We also assume that if two or more players quit at the same price, the tie is broken assigning the object to each player with positive probability.[7]

Theorem 4 *(Necessity of the own effect property). Let the functions $v_1, \dots, v_n$ be regular and increasing at ties. If $v_1, \dots, v_n$ do not satisfy the own effect property for some interior points $\mathbf{s}$ and $\mathbf{s}'$ with $\mathbf{s}' > \mathbf{s}$, then there is a μ such that no μ-efficient equilibrium in undominated strategies exists.*

[6] One can obtain a simple proof of the result using Theorem 7 below, which shows that the single crossing property in Birulin and Izmalkov [1] is stronger than the own effect property, and their necessity result. In that case, one has to assume that gradients are positive.

[7] Papers that deal with necessity have usually assumed this either explicitly or implicitly.

3.3. Related literature

In this section we show that the two properties used by Krishna [9] (Average and Cyclical Crossing Conditions) imply the own-effect property. We also show that under the structure used by Birulin and Izmalkov [1] their "generalized single crossing property" is equivalent to the OEP.

For any $P \subset N$, let I_P denote the vector in $\mathbf{R}^n$ with 1 in the jth coordinate iff $j \in P$ and 0 otherwise and let ∇v_k denote the gradient of v_k.

Definition 5. Say that v satisfies

(a) Krishna's *Average Crossing Condition* (*ACC*) if for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$

$$\sum_{k=1}^{n} \frac{\partial v_k}{\partial s_j} > n \frac{\partial v_i}{\partial s_j}.$$

(b) Krishna's *Cyclical Crossing Condition* (*CCC*) if for all j

$$\frac{\partial v_j}{\partial s_j} > \frac{\partial v_{j+1}}{\partial s_j} \geqslant \frac{\partial v_{j+2}}{\partial s_j} \geqslant \cdots \geqslant \frac{\partial v_{j-1}}{\partial s_j}$$

holds at every $\mathbf{s}$ with $|W(\mathbf{s})| > 1$, where $j + k \equiv (j + k)$ modulo n.

Theorem 5. *If v satisfies the Average Crossing, or the Cyclical Crossing, condition, then it satisfies the OEP.*

As an illustration of the importance of the OEP assumption for auctions, we now show that when there are two players, it is weaker than the Single Crossing Condition: Suppose there are two players; the functions $v = (v_1, v_2)$ satisfy the *Single Crossing Condition* if at any $\mathbf{s}$ such that $v_1(\mathbf{s}) = v_2(\mathbf{s})$

$$\frac{\partial v_i(\mathbf{s})}{\partial s_j} < \frac{\partial v_j(\mathbf{s})}{\partial s_j}.$$

This is the version in Dasgupta and Maskin [3]. In Maskin [10] the inequality is weak, but applies to all $\mathbf{s}$. Since single crossing is necessary for the existence of efficient equilibria in two-player auctions, the OEP is also necessary.

Corollary 6. *With 2 players, OEP is weaker than Single Crossing. Suppose that there are only two players, and that v_1 and v_2 are differentiable. If v satisfies the SC condition or the* (*Maskin*) *Single Crossing, it satisfies the OEP.*

Corollary 6 is a consequence of Theorem 5, as both of Krishna's conditions are equivalent to Single Crossing with two players. We present a direct proof of Corollary 6 because it is simple and instructive.

Proof of Corollary 6. Take any $\mathbf{s}''$ with $|W(\mathbf{s}'')| > 1$ (i.e. $v_1(\mathbf{s}'') = v_2(\mathbf{s}'')$) and an $\mathbf{s}' \geqslant \mathbf{s}''$ such that $s_1' > s_1''$, $s_2' = s_2''$. We will now show that $v_1(s_1', s_2'') \geqslant v_2(s_1', s_2'')$. Let

$$\varepsilon^* = \max\{\varepsilon \in [0, 1]\colon\ v_1\big(\varepsilon\mathbf{s}' + (1 - \varepsilon)\mathbf{s}''\big) \geqslant v_2\big(\varepsilon\mathbf{s}' + (1 - \varepsilon)\mathbf{s}''\big)\}$$

and notice that ε^* is well defined, since 0 belongs to the set over which the maximum is taken. If $\varepsilon^* = 1$, there is nothing to prove, so suppose that the OEP is violated, so that $\varepsilon^* < 1$. Define $\mathbf{s} = \varepsilon^*\mathbf{s}' + (1-\varepsilon^*)\mathbf{s}''$. We then have: $v_1(\mathbf{s}) = v_2(\mathbf{s})$ and for all $\bar{s}_1$ such that $s_1' > \bar{s}_1 > s_1$, $v_1(\bar{s}_1, s_2) < v_2(\bar{s}_1, s_2)$ obtains. This implies that for all $\bar{s}_1 > s_1$

$$v_1(\bar{s}_1, s_2) - v_1(\mathbf{s}) < v_2(\bar{s}_1, s_2) - v_2(\mathbf{s}) \quad \Rightarrow \quad \frac{\partial v_1(\mathbf{s})}{\partial s_1} \leqslant \frac{\partial v_2(\mathbf{s})}{\partial s_1}$$

which contradicts the SC condition, and therefore proves that if SC holds, so does the OEP.

We will now show that if the Maskin Single Crossing holds, so does the OEP. As before, assume $\varepsilon^* < 1$, so that

$$v_1\left(s_1', s_2''\right) < v_2\left(s_1', s_2''\right) \tag{7}$$

and define $\mathbf{s} = \varepsilon^*\mathbf{s}' + (1-\varepsilon^*)\mathbf{s}''$ which implies $v_1(\mathbf{s}) = v_2(\mathbf{s})$. This last equality and Eq. (7) contradict Maskin's Single Crossing since $\forall \bar{s}_1 \in [s_1', s_1]$

$$\frac{\partial v_1(\bar{s}_1, s_2'')}{\partial s_1} \geqslant \frac{\partial v_2(\bar{s}_1, s_2'')}{\partial s_1} \quad \Rightarrow \quad \int_{s_1}^{s_1'} \frac{\partial v_1(\bar{s}_1, s_2'')}{\partial s_1}\, ds_1 \geqslant \int_{s_1}^{s_1'} \frac{\partial v_2(\bar{s}_1, s_2'')}{\partial s_1}\, ds_1$$

$$\Rightarrow \quad v_1\left(s_1', s_2''\right) - v_1\left(s_1, s_2''\right) \geqslant v_2\left(s_1', s_2''\right) - v_2\left(s_1, s_2''\right) \quad \Leftrightarrow \quad v_1\left(s_1', s_2''\right) \geqslant v_2\left(s_1', s_2''\right)$$

as was to be shown. □

We now turn to Birulin and Izmalkov [1]. We show that our Theorem 3 implies their main result. The main assumptions in Birulin and Izmalkov are: regularity (introduced in Eq. (6)), that $\nabla v_j(\mathbf{s}) \geqslant \mathbf{0}$ for all j and $\mathbf{s}$ (which implies increasing at ties), and the following property:

Definition 6. The set of functions v satisfy the *Generalized Single Crossing* property if for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and any $A \subset W(\mathbf{s})$,

$$\max_{j \in A} \mathbf{u}\nabla v_j(\mathbf{s}) \geqslant \mathbf{u}\nabla v_k(\mathbf{s})$$

for all $k \in W(\mathbf{s}) \setminus A$ and any $\mathbf{u}$ such that $u_i > 0$ for $i \in A$ and $u_j = 0$ otherwise.

The next result shows that if one assumes all the conditions of Birulin and Izmalkov, then the GSC and the OEP are equivalent.

Theorem 7. *Suppose that* $\mathbf{s}$ *is drawn from a density,* v*'s are twice differentiable, regular, and* $\nabla v_j(\mathbf{s}) \geqslant \mathbf{0}$ *for all* j *and* $\mathbf{s}$*. Then,* v *satisfies the OEP if and only if it satisfies the GSC.*

Proof. If v satisfies the GSC, it satisfies the OEP. Pick any $\mathbf{s}$ such that $|W(\mathbf{s})| > 1$ and suppose that $\mathbf{s}' \geqslant \mathbf{s}$ and $s_j' > s_j$ only for some $j \in W(\mathbf{s})$. We will now show that $\max_{j:\, s_j' > s_j} v_j(\mathbf{s}') \geqslant \max_{k:\, s_k' = s_k} v_k(\mathbf{s}')$. To obtain a contradiction, suppose that for some player i with $s_i' = s_i$ we have $v_i(\mathbf{s}') = \max_{k:\, s_k' = s_k} v_k(\mathbf{s}') > \max_{j:\, s_j' > s_j} v_j(\mathbf{s}')$. In the equilibrium proposed by Birulin and Izmalkov, all players with $s_i' = s_i$ are inactive at $p = v_j(\mathbf{s})$ for j such that $s_j' > s_j$ (either they had quit before p or quit at p) and so cannot win the auction when types are $\mathbf{s}'$. Since the players with maximum valuations at $\mathbf{s}'$ are inactive, the equilibrium cannot be efficient, which would contradict Proposition 1 in Birulin and Izmalkov (which asserts that, under their assumptions, the proposed equilibrium is efficient). This proves that the OEP is weaker than GSC.

If v satisfies the OEP, it satisfies the GSC. Suppose that v does not satisfy the GSC so that for some $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and some $A \subset W(\mathbf{s})$,

$$\max_{j \in A} \mathbf{u}\nabla v_j(\mathbf{s}) < \mathbf{u}\nabla v_k(\mathbf{s})$$

for some $k \in W(\mathbf{s}) \setminus A$ and some $\mathbf{u}$ such that $u_i > 0$ for $i \in A$ and $u_j = 0$ otherwise. Then, it must be the case that for ε sufficiently small we have $v_k(\mathbf{s} + \varepsilon\mathbf{u}) > \max_{j \in A} v_j(\mathbf{s} + \varepsilon\mathbf{u})$ contradicting the OEP (with $\mathbf{s}' = \mathbf{s} + \varepsilon\mathbf{u}$). $\square$

The following example shows that GSC is not sufficient in the absence of regularity.

Example 1. Suppose two players, 1 and 2, whose signals s_1 and s_2 are drawn independently from a density on [0, 1]. Let

$$z_1(s_1) = (2s_1 - 1)^5 \quad \text{and} \quad z_2(s_1) = \begin{cases} (2s_1 - 1)^3 & s_1 \leqslant \frac{1}{2}, \\ 2(2s_1 - 1)^3 & s_1 \geqslant \frac{1}{2}. \end{cases}$$

It is easy to check that if valuations are $v_i(\mathbf{s}) = s_1 + s_2 + z_i(s_1) + 1$, then all of Birulin and Izmalkov's assumptions are satisfied, except for regularity. Also, there is no efficient equilibrium, since we would need that for all $s_1 < \frac{1}{2}$, $\beta_1(s_1) > \beta_2(s_2)$ for all s_2 and for all $s_1 > \frac{1}{2}$, $\beta_1(s_1) < \beta_2(s_2)$ for all s_2. But then, when player 1 has a signal of 1, he is strictly better off bidding as if he had a signal of $1/4$, showing that there is no efficient equilibrium.

4. Final remarks

We have used similar "relative sensitivity" assumptions on a system of equations to obtain results in two seemingly different applications. Intuitively we require that a variable be associated to each function so that the own effect is stronger than the cross effect. There are alternative formalizations of this intuition. We used three different ones in the paper, the dominant-effect, the adapted dominant-effect, and the own effect property. The adapted dominant-effect is simply a version of the dominant-effect adapted to non-scalar variables, for the non-square Stolper–Samuelson theorem.

The own effect property is weaker than the dominant-effect property. In trying to establish the efficiency of the English auction we looked for the weakest condition that would yield the result. Indeed, we also show that the own-effect property is necessary if one is willing to assume a regularity condition.

Both properties, however, have much in common. Suppose there are three functions with three variables and that each variable s_i has a stronger influence on v_i than the other variables do. Both properties rule out the possibility that if s_1 and s_2 increase, the change in v_3 might dominate the changes in v_1 and v_2.

Appendix A. Proof of results on English auctions

The proof of Theorem 1 is based on the construction of an equilibrium with certain properties. This kind of equilibrium was previously used in Milgrom and Weber [12], Maskin [10], Krishna [9] and Birulin and Izmalkov [1]. It is based on the following simple idea. Since exits in

English auctions are public, one player's quitting conveys information to the other players about the quitter's signal. Suppose there is an increasing function $\sigma(p)$ mapping prices into profiles of signals such that $v_i(\sigma(p)) = v_j(\sigma(p)) = p$ for all i and j. Suppose that no player has quit, and the price is p. Then, in the proposed equilibrium player i stays in the auction as long as $s_i > \sigma_i(p)$ and quits when $s_i = \sigma_i(p)$. Therefore, when a player quits, his signal $s_i = \sigma_i(p)$ becomes known. This is a reasonable strategy since, as long as nobody quits, players know that $\mathbf{s} \geqslant \sigma(p)$ and therefore $v_i(\mathbf{s}) \geqslant p$ for all i. In any sub-auction in which the set of active players is B, let us call $\mathbf{y}^{N\setminus B}$ the vector of known signals of the players who have already quit. The informal description of the strategies that will be used in the efficient ex-post equilibrium are the following:

- in the empty history, player i remains in the auction as long as $b \geqslant s_i > \sigma_i(p)$ (the profile of signals $\mathbf{0}$ satisfies (a) and (b) of Lemma A.1 below, so the function σ exists); all players know this; player i drops at the lowest price p such that $s_i = \sigma_i(p)$; let the price of the first drop be p^1, let i^* be the player who drops at p^1 and at the time of his drop, player i^*'s signal becomes known, so let $y_{i^*} = \sigma_{i^*}(p^1)$;
- let $A = N\setminus\{i^*\}$ and $\mathbf{y}^{N\setminus A} \equiv y_{i^*}$ and notice that since $\sigma(p^1)$ satisfies $v_j(\sigma(p^1)) = p$ for all j, the profile $\mathbf{y}^A = \sigma_{-i^*}(p^1)$ satisfies the conditions of Lemma A.1, so that a function $\sigma^{\mathbf{y}^{N\setminus A}}$ satisfying (i)–(iii) in that lemma exists. Then, player $j \in A$ remains in the auction as long as $s_j \geqslant \sigma^{\mathbf{y}^{N\setminus A}}(p)$, and drops at the lowest p such that $s_j = \sigma^{\mathbf{y}^{N\setminus A}}(p)$;
- the process continues in this fashion.

The formal description of the strategies just mentioned is as follows: in a subgame in which types $\mathbf{y}^{N\setminus A}$ are known and active players are A, $\beta_i^A(s_i, \mathbf{y}^{N\setminus A}) = \beta_i^{\mathbf{y}^{N\setminus A}}(s_i) = \min\{p\colon \sigma^{\mathbf{y}^{N\setminus A}}(p) \geqslant s_i\}$. Notice that since σ is continuous and weakly increasing, β is strictly increasing and well defined.

The following lemma proves the existence of a σ function as described above for any (relevant) sub-auction. For any set $A \subseteq N$, any player $i \in A$, and any $\mathbf{y}$, let $V_i^{\mathbf{y}^{N\setminus A}} : [0, b]^{|A|} \to \mathbf{R}$ be defined by $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = v_i(\mathbf{s}, \mathbf{y}^{N\setminus A})$.

Lemma A.1. *Fix any $B \subseteq N$, with $|B| > 1$, and fix a profile of types $\mathbf{y}^{N\setminus B}$ such that there exists $\mathbf{y}^B \neq \mathbf{b}$ for which for all $i \in B$, $y_i < b$ implies $v_i(\mathbf{y}) = \max_{j\in N} v_j(\mathbf{y})$. If v is increasing at ties and satisfies the OEP, there exists a $p_{\mathbf{y}}^B > \max_i v_i(\mathbf{y})$ and a weakly increasing function $\sigma^{\mathbf{y}^{N\setminus B}} : [\max_i v_i(\mathbf{y}), p_{\mathbf{y}}^B] \to \prod_{i\in B}[y_i, b]$ mapping prices into types of active players, such that:*

(i) *$\sigma_j^{\mathbf{y}^{N\setminus B}}(p_{\mathbf{y}}^B) = b$ for some j with $y_j < b$ and for all $i \in B$, $p = p_{\mathbf{y}}^B$ and $y_i < b$ imply the break even condition*

$$V_i^{\mathbf{y}^{N\setminus B}}\big(\sigma^{\mathbf{y}^{N\setminus B}}(p)\big) = p \tag{A.1}$$

holds;

(ii) *for all $p < p_{\mathbf{y}}^B$, if $y_i < b$ then $\sigma_i^{\mathbf{y}^{N\setminus B}}(p) < b$ and the break even condition* (A.1) *hold*;

(iii) *for all $p \leqslant p_{\mathbf{y}}^B$, and all $k \in N$, $v_k(\sigma^{\mathbf{y}^{N\setminus B}}(p), \mathbf{y}^{N\setminus B}) \leqslant p$.*

The proof of Lemma A.1 is based on the following lemma.

Lemma A.2. *Fix any $A \subseteq N$, with $|A| > 1$, and fix a profile of types $\mathbf{y}^{N\backslash A}$ such that there exists a $\mathbf{y}^A$ for which*:

(a) *for all $i, j \in A$, $v_i(\mathbf{y}) = v_j(\mathbf{y})$ and $y_i, y_j < b$*;
(b) *for all $k \notin A$, and $i \in A$, $v_k(\mathbf{y}) \leqslant v_i(\mathbf{y})$.*

If v is increasing at ties and satisfies the OEP, there exists a $p_{\mathbf{y}}^A > v_i(\mathbf{y}) = V_i^{\mathbf{y}^{N\backslash A}}(\mathbf{y}^A)$ (for $i \in A$) and a weakly increasing function $\sigma^{\mathbf{y}^{N\backslash A}} : [V_i^{\mathbf{y}^{N\backslash A}}(\mathbf{y}^A), p_{\mathbf{y}}^A] \to \prod_{i\in A}[y_i, b]$ mapping prices into types of active players, such that:

(i) *$\sigma_j^{\mathbf{y}^{N\backslash A}}(p_{\mathbf{y}}^A) = b$ for some j, and for all $i \in A$, $p = p_{\mathbf{y}}^A$ implies that the condition* (A.1) *holds*;
(ii) *for all $p < p_{\mathbf{y}}^A$, $\sigma^{\mathbf{y}^{N\backslash A}}(p) \ll \mathbf{b} = (b, \dots, b)$ and the break even condition* (A.1) *holds for all $i \in A$*;
(iii) *for all $p \leqslant p_{\mathbf{y}}^A$, and all $k \in N$, $v_k(\sigma^{\mathbf{y}^{N\backslash A}}(p), \mathbf{y}^{N\backslash A}) \leqslant p$.*

Proof. Fix any A and $\mathbf{y}$ that satisfy conditions (a) and (b). Let $(b, \mathbf{y}_{-i}^A)$ denote the vector $\mathbf{y}^A$ with the ith component replaced by a b. Since $V_i^{\mathbf{y}^{N\backslash A}}$ is strictly increasing in s_i and $y_i < b$ (by (a)) we get for all i, $V_i^{\mathbf{y}^{N\backslash A}}(\mathbf{y}^A) < V_i^{\mathbf{y}^{N\backslash A}}(b, \mathbf{y}_{-i}^A)$.

Defining a non-empty set X. For any $i \in A$, let $\pi = [V_i^{\mathbf{y}^{N\backslash A}}(\mathbf{y}^A), \min_i V_i^{\mathbf{y}^{N\backslash A}}(b, \mathbf{y}_{-i}^A)]$. Let

$$Y = \left\{(P, \sigma)\colon\ V_i^{\mathbf{y}^{N\backslash A}}\big(\mathbf{y}^A\big) = v_i(\mathbf{y}) \in P \subset \pi,\ \sigma : P \to \prod_A \big[y_i^A, b\big]\right\}$$

and

$$X = \big\{(P, \sigma) \in Y\colon\ \sigma \text{ weakly increasing},\ \sigma\big(v_i(\mathbf{y})\big) = \mathbf{y}^A,\ V_i^{\mathbf{y}^{N\backslash A}}\big(\sigma(p)\big) = p,\ \forall (i, p) \in A \times P\big\}.$$

Notice that by condition (a) $P = \{p\colon\ p = v_i(\mathbf{y}) \text{ for some } i \in A\}$ is a singleton and the function σ defined by $\sigma(v_i(\mathbf{y})) = \mathbf{y}^A$ satisfies $V_i^{\mathbf{y}^{N\backslash A}}(\sigma(p)) = p$. Therefore, X is non-empty.

Defining a partial order on X. Define a partial order on X by $(P', \sigma') \succcurlyeq (P, \sigma)$ if and only if $P' \supseteq P$ and $\sigma'(p) = \sigma(p)$ for all $p \in P$.

Showing that every chain in X has an upper bound. Take any totally ordered set (a chain) $\{(P_\alpha, \sigma_\alpha)\}_\alpha$ in X and define $P \equiv \bigcup_\alpha P_\alpha$ and $\sigma : P \to \prod_A [y_i^A, b]$ through $\sigma(p) = \sigma_\alpha(p)$ for any α such that $p \in P_\alpha$. Notice that the definition of σ does not depend on the specific α chosen, since if p belongs to two different P_α and $P_{\alpha'}$, we still get $\sigma_\alpha(p) = \sigma_{\alpha'}(p)$. I will first show that $(P, \sigma) \in X$, and then that (P, σ) is an upper bound for $\{(P_\alpha, \sigma_\alpha)\}_\alpha$.

It is easy to check that σ is weakly increasing. Also, for any $p \in P$, there is some α for which: $p \in P_\alpha$ and $\sigma_\alpha(p) = \sigma(p)$. Then, since $(P_\alpha, \sigma_\alpha) \in X$, we get

$$V_i^{\mathbf{y}^{N\backslash A}}\big(\sigma_\alpha(p)\big) = p \quad \Rightarrow \quad V_i^{\mathbf{y}^{N\backslash A}}\big(\sigma(p)\big) = p$$

showing that $(P, \sigma) \in X$.

To see that (P, σ) is an upper bound, note that for any α we have $P \supseteq P_\alpha$ and $\sigma(p) = \sigma_\alpha(p)$ for all $p \in P_\alpha$.

Showing that the maximal element implied by Zorn's lemma must have $P = \pi$. Zorn's lemma then ensures that there exists a maximal element (P^M, σ^M) in X. We now show that $P^M = \pi$. Suppose $p' \notin P^M$, notice that we have $v_i(\mathbf{y}) \in P^M$ and $v_i(\mathbf{y})$ is a lower bound for P^M, so $\{\widetilde{p} \in P^M\colon\ \widetilde{p} < p'\}$ is non-empty, so define $p_* = \sup_{\widetilde{p}}\{\widetilde{p} \in P^M\colon\ \widetilde{p} < p'\}$. If there is some $p \in P^M$ such that $p > p'$ let

$$p^* = \inf_{\widetilde{p}}\{\widetilde{p} \in P^M\colon\ \widetilde{p} > p'\}.$$

Case A, $p_* \notin P^M$. Consider first the case in which $p_* \notin P^M$. We set $P' = P^M \cup \{p_*\}$ and letting $\{p_n\}$ be an increasing sequence in P^M that converges to p_*, define σ' on P' through

$$\sigma'(p) = \begin{cases} \sigma'(p) = \sigma^M(p) & \text{for all } p \neq p_*, \\ \sigma'(p_*) = \lim_n \sigma^M(p_n). \end{cases}$$

Since σ^M is increasing, the limit is well defined. Moreover, it is easy to check that σ' is increasing. For all $p \in P^M$, we already know that $V_i^{\mathbf{y}^{N\setminus A}}(\sigma'(p)) = V_i^{\mathbf{y}^{N\setminus A}}(\sigma^M(p)) = p$ holds, and for p_*, we also have that, by continuity of $V_i^{\mathbf{y}^{N\setminus A}}$,

$$V_i^{\mathbf{y}^{N\setminus A}}\big(\sigma'(p_*)\big) = V_i^{\mathbf{y}^{N\setminus A}}\Big(\lim_n \sigma^M(p_n)\Big) = \lim_n V_i^{\mathbf{y}^{N\setminus A}}\big(\sigma^M(p_n)\big) = \lim_n p_n = p_*$$

establishing that $(P', \sigma') \in X$. Since $(P', \sigma') \succ (P^M, \sigma^M)$ by construction, this contradicts (P^M, σ^M) being maximal.

Case B, $p_* \in P^M$ and $\exists p \in P^M$ such that $p > p'$. Consider now the case in which $p_* \in P^M$, so that $p_* < p'$. If there is some $p \in P^M$ such that $p > p'$, one can follow the same steps as in Case A to discard the case in which $p^* \notin P^M$, so assume that $p^* \in P^M$. Let $\underline{\mathbf{s}} = \sigma^M(p_*)$, and $\bar{\mathbf{s}} = \sigma^M(p^*)$ and fix any p with

$$p_* < p < \min V_i^{\mathbf{y}^{N\setminus A}}\big(\bar{s}_i, \sigma_{-i}^M(p_*)\big) \leqslant p^*. \tag{A.2}$$

Assume, without loss of generality, that $\bar{s}_i > \underline{s}_i$ for all i (when they are equal, the signal of player i just becomes a fixed "parameter" in the V functions, and thus plays no role).

Let $g : \mathbf{R} \to (-1, 1)$ be any strictly decreasing function with $g(0) = 0$. For $i \in A$, let

$$h_i(\mathbf{s}) = \begin{cases} s_i + g(V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p)(s_i - \underline{s}_i) & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) > p, \\ s_i & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = p, \\ s_i + g(V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p)(\bar{s}_i - s_i) & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) < p. \end{cases}$$

The function

$$h : \prod_A [\underline{s}_i, \bar{s}_i] \to \prod_A [\underline{s}_i, \bar{s}_i]$$

satisfies hypothesis of Brouwer, so there is a fixed point $\mathbf{s}^f$. We now show that $\forall i$,

$$V_i^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) = p. \tag{A.3}$$

(1) Suppose that for some i, $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) > p$. Then we get $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) - p > 0$, and since $h_i(\mathbf{s}^f) = s_i^f$, we must have $s_i^f = \underline{s}_i$ (otherwise, $g(V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) - p)$ would be subtracting something from s_i^f). We then get $V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}_i, \mathbf{s}_{-i}^f) > p$ and since Eq. (A.2) ensures

$$V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) = V_i^{\mathbf{y}^{N\setminus A}}\big(\sigma(p_*)\big) = p_*,$$

we must have $s_j^f > \underline{s}_j$ for some j. Let k be the player with $s_k^f > \underline{s}_k$ for whom $V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) = \max_{i:\, s_i^f > \underline{s}_i} V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)$. By applying the OEP we see that for player i with $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) > p$ and $s_i^f = \underline{s}_i$,

$$V_k^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) \geqslant \max_{j:\, s_j^f = \underline{s}_j} V_j^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) \geqslant V_i^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) > p.$$

Then, player k is such that $V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) > p$, but since

$$h_k\big(\mathbf{s}^f\big) = s_k^f = s_k^f + g\big(V_k^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) - p\big)\big(s_k^f - \underline{s}_k\big)$$

with $g(V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) - p) < 0$ which contradicts $\mathbf{s}^f$ being a fixed point.

(2) If $V_m^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) < p$, for some m, then, since $h_m(\mathbf{s}^f) = s_m^f$, we must have $s_m^f = \bar{s}_m$, because otherwise $g(V_m^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) - p)$ would be adding something strictly positive to s_m^f. Because we can use that v is increasing at ties with $\mathbf{s}' = (\bar{s}_m, \mathbf{s}_{-m}^f, \mathbf{y}^{N\setminus A})$ and $\mathbf{s} = (\sigma(p_*), \mathbf{y}^{N\setminus A})$, we obtain

$$\begin{aligned} p > V_m^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big) &= V_m^{\mathbf{y}^{N\setminus A}}\big(\bar{s}_m, \mathbf{s}_{-m}^f\big) \geqslant V_m^{\mathbf{y}^{N\setminus A}}(\bar{s}_m, \underline{\mathbf{s}}_{-m}) = V_m^{\mathbf{y}^{N\setminus A}}\big(\bar{s}_m, \sigma_{-m}(p_*)\big) \\ &\geqslant \min V_i^{\mathbf{y}^{N\setminus A}}\big(\bar{s}_i, \sigma_{-i}(p_*)\big) > p \end{aligned}$$

which is a contradiction. That is, we had chosen a small p, so that a large increase in the signal of m from $\underline{s}_m$ to $\bar{s}_m$ increases $V_m^{\mathbf{y}^{N\setminus A}}$ above p.

Items (1) and (2) have established that $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) = p$ for all i, so that $P' = P^M \cup \{p\}$ and

$$\sigma'(\widetilde{p}) = \begin{cases} \sigma'(\widetilde{p}) = \sigma^M(\widetilde{p}) & \text{for all } \widetilde{p} \neq p, \\ \sigma'(p) = \mathbf{s}^f \end{cases}$$

satisfy $(P', \sigma') \succ (P^M, \sigma^M)$ which contradicts (P^M, σ^M) being maximal. We conclude that $P^M = \pi$, and that σ^M maps $\pi = [V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A), \min_i V_i^{\mathbf{y}^{N\setminus A}}(b, \mathbf{y}_{-i}^A)]$ into $\prod_A [y_i^A, b]$, is increasing and $V_i^{\mathbf{y}^{N\setminus A}}(\sigma^M(p)) = p$, $\forall i \in A$, $\forall p \in \pi$.

Case C, $p_* \in P^M$ and $\nexists p \in P^M$ such that $p > p'$. Recall $\underline{s} = \sigma(p_*)$ and fix any p with

$$p_* = V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) < p \leqslant \min V_i\big(b, \mathbf{y}_{-i}^A\big). \tag{A.4}$$

Let $g : \mathbf{R} \to (-1, 1)$ be any strictly decreasing function with $g(0) = 0$. For $i \in A$, let

$$h_i(\mathbf{s}) = \begin{cases} s_i + g(V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p)(s_i - \underline{s}_i) & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) > p, \\ s_i & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) = p, \\ s_i + g(V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) - p)(b - s_i) & \text{if } V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}) < p. \end{cases}$$

The function h has a fixed point $\mathbf{s}^f$, so we will show that for all i, $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) = p$.

(1) Suppose that for some i, $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f) > p$, so that $s_i^f = \underline{s}_i$. We then get $V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}_i, \mathbf{s}_{-i}^f) > p$ and since $V_i^{\mathbf{y}^{N\setminus A}}(\underline{s}) < p$, we must have $s_j^f > \underline{s}_j$ for some j. Let k be the player with $s_k^f > \underline{s}_k$ for

whom $V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)=\max_{i:\, s_i^f>\underline{s}_i} V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)$. By applying the OEP we see that for player i with $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)>p$ and $s_i^f=\underline{s}_i$,

$$V_k^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big)\geqslant \max_{j:\, s_j^f=\underline{s}_j} V_j^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big)\geqslant V_i^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big)>p.$$

Then, player k is such that $V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)>p$, but since $s_k^f>\underline{s}_k$,

$$h_k\big(\mathbf{s}^f\big)=s_k^f=s_k^f+g\big(V_k^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big)-p\big)\big(s_k^f-\underline{s}_k\big)$$

with $g(V_k^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)-p)<0$ which contradicts $\mathbf{s}^f$ being a fixed point.

(2) If $V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)<p$, for some i, then, since $h_i(\mathbf{s}^f)=s_i^f$, we must have $s_i^f=b$. Then, using the choice of p in Eq. (A.4) and that v is increasing at ties, with $\mathbf{s}'=(b,\mathbf{s}_{-i}^f,\mathbf{y}^{N\setminus A})$ and $\mathbf{s}=\mathbf{y}$, we obtain

$$p>V_i^{\mathbf{y}^{N\setminus A}}\big(\mathbf{s}^f\big)=V_i^{\mathbf{y}^{N\setminus A}}\big(b,\mathbf{s}_{-i}^f\big)\geqslant V_i^{\mathbf{y}^{N\setminus A}}\big(b,\mathbf{y}_{-i}^A\big)\geqslant \min V_i^{\mathbf{y}^{N\setminus A}}\big(b,\mathbf{y}_{-i}^A\big)\geqslant p$$

which is a contradiction.

Items (1) and (2) have established for all i, so that $P'=P^M\cup\{p\}$ and

$$\sigma'(\widetilde{p})=\begin{cases}\sigma'(\widetilde{p})=\sigma^M(\widetilde{p}) & \text{for all } \widetilde{p}\neq p,\\ \sigma'(p)=\mathbf{s}^f\end{cases}$$

satisfy $(P',\sigma')\succ(P^M,\sigma^M)$ which contradicts (P^M,σ^M) being maximal. We conclude that $P^M=\pi$, and that σ^M maps $\pi=[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A),\min_i V_i^{\mathbf{y}^{N\setminus A}}(b,\mathbf{y}_{-i}^A)]$ into $\prod_A[y_i^A,b]$, is increasing and $V_i^{\mathbf{y}^{N\setminus A}}(\sigma^M(p))=p$, $\forall i\in A$, $\forall p\in\pi$.

So far we have established that for all p in $[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A),\min_i V_i^{\mathbf{y}^{N\setminus A}}(b,\mathbf{y}_{-i}^A)]$ there exists of a profile of signals $\sigma^{\mathbf{y}^{N\setminus A}}(p)\equiv\mathbf{s}^f$ such that $V_i^{\mathbf{y}^{N\setminus A}}(\sigma^{\mathbf{y}^{N\setminus A}}(p))=V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{s}^f)=p$ for all i, for all $p\leqslant\min V_i(b,\mathbf{y}_{-i}^A)$, and $\sigma^{\mathbf{y}^{N\setminus A}}$ is increasing. Since $\mathbf{y}$ and A are fixed throughout the proof, we will let $\sigma(p)$ stand for $\sigma^{\mathbf{y}^{N\setminus A}}(p)$ and V_i for $V_i^{\mathbf{y}^{N\setminus A}}$.

Let $p^1=\min V_i(b,\mathbf{y}_{-i}^A)$ and fix $\mathbf{s}^1=\sigma(p^1)$. If $s_i^1=b$ for some i, the proof is complete by letting $p_{\mathbf{y}}^A=p^1$ since for all $p<p^1$ we have that $\sigma(p)\ll b$, for if $\sigma_i(p)$ was equal to b, we would get the following contradiction

$$p=V_i\big(\sigma(p)\big)\geqslant\min V_i\big(\sigma(p)\big)\geqslant\min V_i\big(b,\mathbf{y}_{-i}^A\big)=p_1>p.$$

So assume $s_i^1<b$ for all i. Then, we have that

$$p^1=\min V_i\big(b,\mathbf{y}_{-i}^A\big)=V_i\big(\sigma\big(p^1\big)\big)=V_i\big(\mathbf{s}^1\big)$$

and $s_i^1<b$ imply that $p^1<\min V_i(b,\mathbf{s}_{-i}^1)\equiv p^2$. Fix any $p^1<p\leqslant p^2$. We can now repeat exactly the same steps as we have done so far (with $\mathbf{s}^1$ in place of $\mathbf{y}^A$) and show that in the domain $[V_i^{\mathbf{y}^{N\setminus A}}(\mathbf{y}^A),\min_i V_i(b,\mathbf{s}_{-i}^1)]$ one has an increasing function $\sigma(\cdot)$ such that $V_i(\sigma(p))=p$ for all i. Fix any $\mathbf{s}^2=\sigma(p^2)$, and notice again that if $\sigma_i(p^2)=b$ for some i, the proof is complete by letting $p_{\mathbf{y}}^A=p^2$.

Continuing in this fashion, we get an increasing sequence of $\mathbf{s}^t$ and p^t with the properties that for all i,

$$V_i\big(\mathbf{s}^t\big)=p^t<p^{t+1}=\min_i V_i\big(b,\mathbf{s}_{-i}^t\big).$$

In the limit p^{∞}, $\mathbf{s}^{\infty}$ we obtain for all i

$$V_i\left(\mathbf{s}^{\infty}\right) = p^{\infty} = \min_i V_i\left(b, \mathbf{s}_{-i}^{\infty}\right)$$

and so, for some i, $V_i(\mathbf{s}^{\infty}) = p^{\infty} = V_i(b, \mathbf{s}_{-i}^{\infty})$. Since V_i is increasing in s_i this means that $s_i^{\infty} = b$, so that we can set $p_{\mathbf{y}}^{A} = p^{\infty}$. This completes the proof of (i) and (ii).

To establish (iii) set $\mathbf{s}' = (\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A})$ and $\mathbf{s} = \mathbf{y}$. If $\mathbf{s}' = \mathbf{s}$ conditions (a) and (b) yield the desired result, so assume $\mathbf{s}' \neq \mathbf{s}$. Note that: $k \in A$ implies $p = v_k(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A})$; $k \notin A$ implies that $s'_k = s_k$ so that the OEP ensures

$$p = \max_{k:\ s'_k > s_k} v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right) \geqslant \max_{k:\ s'_k = s_k} v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right) \geqslant v_k\left(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}\right)$$

for all $k \notin A$ as was to be shown. $\square$

The previous lemma establishes the existence of a σ function that maps prices into signals, the resulting profile of signals being the "presumption" that other players will have about a players' signal, if he quits at a certain price. The set A is the set of "active" players at a certain moment, and the profile of signals $\mathbf{y}$ is decomposed in the set of signals of inactive players $\mathbf{y}^{N\setminus A}$ and the set of signals such that all active players have signals greater than $\mathbf{y}^{A}$. Lemma A.1 describes the presumption of other players about a certain player's signal, when he should have quit, but he did not (in the sense that his presumed signal is b, but he did not quit). The difference with the previous lemma is that we allow some elements of $\mathbf{y}^{B}$ to be equal to b (whereas in Lemma A.2 we had $y_i^{B} < b$ for all i in B).

Proof of Lemma A.1. Let B and $\mathbf{y}$ be as in the statement of this lemma. Consider first the case in which $y_k < b$ for $k = i, j \in B, i \neq j$. Defining $A = B\setminus\{j \in B\colon\ y_j = b\}$ and applying Lemma A.2 yields the desired result. So assume there is a unique $i \in B$ such that $y_i < b$. Let $p_{\mathbf{y}}^{B} = v_i(b, \mathbf{y}_{-i})$ and let $v_i^{-1}(p; \mathbf{y}_{-i})$ be the "inverse" of v_i, defined by

$$v_i\left(v_i^{-1}(p; \mathbf{y}_{-i}), \mathbf{y}_{-i}\right) \equiv p.$$

Then, it is easy to check that $\sigma^{\mathbf{y}^{N\setminus B}}$ defined by

$$\sigma_j^{\mathbf{y}^{N\setminus B}}(p) = \begin{cases} b, & j \in B\setminus\{i\}, \\ v_i^{-1}(p; \mathbf{y}_{-i}), & j = i \end{cases}$$

satisfies conditions (i) and (ii). To check condition (iii), two cases must be considered.

(I) If $|W(\mathbf{y})| > 1$, we have that for $\mathbf{s} = \mathbf{y}$, and

$$\mathbf{s}' = \left(\sigma_i^{\mathbf{y}^{N\setminus B}}(p), \mathbf{y}_{-i}\right) = \left(\sigma_i^{\mathbf{y}^{N\setminus B}}(p), b, \ldots, b, \mathbf{y}^{N\setminus B}\right) = \left(\sigma^{\mathbf{y}^{N\setminus B}}(p), \mathbf{y}^{N\setminus B}\right)$$

the OEP implies that since i is the only player for which $s'_i > s_i$, for all p,

$$p = v_i(\mathbf{s}') = \max_{j:\ s'_j > s_j} v_j(\mathbf{s}') \geqslant \max_{j \neq i} v_j(\mathbf{s}') = \max_{j \neq i} v_j\left(\sigma^{\mathbf{y}^{N\setminus B}}(p), \mathbf{y}^{N\setminus B}\right)$$

as was to be shown.

(II) If $|W(\mathbf{y})| = 1$, we have that for $p_* = \max_j v_j(\mathbf{y}) = v_i(\mathbf{y})$,

$$v_i\left(\sigma^{\mathbf{y}^{N\setminus B}}(p_*), \mathbf{y}^{N\setminus B}\right) = \max_{j \in N} v_j(\mathbf{y}) > \max_{j \neq i} v_j(\mathbf{y}) = \max_{j \neq i} v_j\left(\sigma^{\mathbf{y}^{N\setminus B}}(p_*), \mathbf{y}^{N\setminus B}\right). \tag{A.5}$$

Suppose that contrary to what we want to show, there was some $\overline{p}$ such that for some $j \neq i$

$$v_j\big(\sigma^{\mathbf{y}^{N\setminus B}}(\overline{p}), \mathbf{y}^{N\setminus B}\big) > \overline{p} = v_i\big(\sigma^{\mathbf{y}^{N\setminus B}}(\overline{p}), \mathbf{y}^{N\setminus B}\big). \tag{A.6}$$

Given Eqs. (A.5) and (A.6), continuity of $\sigma^{\mathbf{y}^{N\setminus B}}(p)$ (ensured by construction) and Bolzano's Theorem, there exists a p^* such that $\max_{j\neq i} v_j(\sigma^{\mathbf{y}^{N\setminus B}}(p^*), \mathbf{y}^{N\setminus B}) = v_i(\sigma^{\mathbf{y}^{N\setminus B}}(p^*), \mathbf{y}^{N\setminus B})$. Then, letting $\mathbf{s}' = (\sigma_i^{\mathbf{y}^{N\setminus B}}(\overline{p}), \mathbf{y}_{-i})$ and $\mathbf{s} = (\sigma_i^{\mathbf{y}^{N\setminus B}}(p^*), \mathbf{y}_{-i})$ the OEP implies

$$v_i(\mathbf{s}') \geqslant \max_{k\neq i} v_k(\mathbf{s}') \geqslant v_j(\mathbf{s}') \quad \Leftrightarrow \quad v_i\big(\sigma_i^{\mathbf{y}^{N\setminus B}}(\overline{p}), \mathbf{y}_{-i}\big) \geqslant v_j\big(\sigma^{\mathbf{y}^{N\setminus B}}(\overline{p}), \mathbf{y}^{N\setminus B}\big)$$

which contradicts (A.6), and therefore completes the proof. □

The next lemma gives the connection between one set of functions σ^B and the set of functions σ^A when $A = B\setminus\{l\}$ for some $l \in B$. This gives the relation between the bidding strategies in a sub-auction with active players B, and the one that follows after player l has dropped out. If various players drop out at the same price, one only needs to apply the lemma repeatedly at the price of the drops ($\widetilde{p}$ in the lemma).

Lemma A.3. *Fix any $B \subseteq N$, with $|B| > 2$, and fix a set of types $\mathbf{y}^{N\setminus B}$ such that there exists a $\mathbf{y}^B \neq \mathbf{b}$ for which for all $i \in B$, $y_i < b$ implies $v_i(\mathbf{y}) = \max_{j\in N} v_j(\mathbf{y})$. Assume that v is increasing at ties and satisfies the OEP, and fix a $p_{\mathbf{y}}^B$ and $\sigma^{\mathbf{y}^{N\setminus B}}$ as in the statement of Lemma* A.1. *Fix any $l \in B$ and let $A = B\setminus\{l\}$. For any $\widetilde{p} \leqslant p_{\mathbf{y}}^B$, if $\mathbf{s}^B = \sigma^{\mathbf{y}^{N\setminus B}}(\widetilde{p})$ then for $z \equiv (\mathbf{y}^{N\setminus B}, s_l)$ there exists $p_z^A \geqslant v_i(\mathbf{s}^B, \mathbf{y}^{N\setminus B}) = V_i^z(\mathbf{s}^A)$ (for i with $y_i < b$) and a weakly increasing function $\sigma^z : [V_i^z(\mathbf{s}^A), p_z^A] \to \prod_{i\in A}[s_i, b]$ mapping prices into types of active players, such that*:

(i) *$\sigma_j^z(p_z^A) = b$ for some j with $y_j < b$ and for all $i \in A$, $p = p_{\mathbf{y}}^A$ and $y_i < b$ imply the break even condition*

$$V_i^z\big(\sigma^z(p)\big) = p; \tag{A.7}$$

(ii) *for all $p < p_z^A$, if $y_i < b$ then $\sigma_i^z(p) < b$ and the break even condition* (A.7) *holds for all $i \in A$;*

(iii) *for all $p \leqslant p_{\mathbf{y}}^A$, and all $k \in N$, $v_k(\sigma^{\mathbf{y}^{N\setminus A}}(p), \mathbf{y}^{N\setminus A}) \leqslant p$;*

(iv) *for all $j \in A$*

$$\sigma_j^z(\widetilde{p}) = \sigma_j^{\mathbf{y}^{N\setminus B}}(\widetilde{p}).$$

Proof. Items (i), (ii) and (iii) follow as a direct application of Lemma A.1. Then, item (iv) follows because for all i, $\sigma_i^z(\widetilde{p}) \geqslant s_i^A$, and if $\sigma_j^z(\widetilde{p}) > s_j^A$ we would get (using $\mathbf{s}' = (s_j^A, \sigma_{-j}^z(p), z)$ and $\mathbf{s} = (\mathbf{s}^A, z)$ and that v is increasing at ties)

$$\begin{aligned}\widetilde{p} &= V_j^z\big(\sigma^z(\widetilde{p})\big) > V_j^z\big(s_j^A, \sigma_{-j}^z(p)\big) \geqslant V_j^z\big(\mathbf{s}^A\big)\\ &= v_j\big(\mathbf{s}^B, \mathbf{y}^{N\setminus B}\big) = V_j^{\mathbf{y}^{N\setminus B}}\big(\mathbf{s}^B\big) = V_j^{\mathbf{y}^{N\setminus B}}\big(\sigma^{\mathbf{y}^{N\setminus B}}(\widetilde{p})\big) = \widetilde{p}\end{aligned}$$

which is a contradiction. □

We now show that the σ function is continuous.

Lemma A.4. *For every A and $\mathbf{y}^{N\setminus A}$ satisfying the conditions of Lemma* A.1*, the function $\sigma^{\mathbf{y}^{N\setminus A}}$ is continuous.*

Proof. Suppose that σ is discontinuous at p^*. It must be either not continuous from the right, or from the left, so assume without loss of generality that it is discontinuous from the left: there is an ε such that for all δ there is some p with $p^* - p < \delta$ but $\sigma(p^*) - \sigma(p) \geqslant \varepsilon$ (we have used σ non-decreasing). Fix then $\delta_1 = 1$ and $p_1 < p^*$ such that $p^* - p_1 < \delta_1$ but $\sigma(p^*) - \sigma(p_1) \geqslant \varepsilon$. Pick then, by induction, $\delta_n = (p^* - p_{n-1})/2$ and $p^* - p_n < \delta_n$ but $\sigma(p^*) - \sigma(p_n) \geqslant \varepsilon$. We then obtain: $p_n \to p^*$, p_n is increasing, $\sigma(p_n)$ is increasing and therefore has a limit (since its bounded above by b) $\mathbf{s}^\infty$ and $\mathbf{s}^\infty \neq \sigma(p^*)$, $\mathbf{s}^\infty \leqslant \sigma(p^*)$.

Since for all n and for all i, $V_i(\sigma(p_n)) = p_n$ we obtain by continuity of V_i,

$$p^* = \lim p_n = \lim V_i\big(\sigma(p_n)\big) = V_i\big(\lim \sigma(p_n)\big) = V_i\big(\mathbf{s}^\infty\big).$$

But then, $\mathbf{s}^\infty \neq \sigma(p^*)$ and $\mathbf{s}^\infty \leqslant \sigma(p^*)$ imply that for some i, $s_i^\infty < \sigma_i(p^*)$. This, in turn, means that since V_i is strictly increasing in s_i and increasing at ties (at $\mathbf{s}^\infty$ all are tied), $V_i(\mathbf{s}^\infty) < V_i(\sigma(p^*)) = p^*$. This is a contradiction, and shows that σ is continuous. $\square$

Proof of Theorem 3. Ex Post. We will prove that the profile of strategies that in any auction with active players A and signals of inactive players $\mathbf{y}^{N\setminus A}$ calls for a player with signal s_i to quit at a price $\beta_i^{\mathbf{y}^{N\setminus A}}(s_i) = \min\{p\colon \sigma^{\mathbf{y}^{N\setminus A}}(p) \geqslant s_i\}$, for σ as in Lemma A.1, is an ex post equilibrium. We will then show that it is also efficient.

The first part of the proof (ex-post equilibrium) follows Krishna [9] Lemma 1 closely, but does not use the fact that σ is unique or strictly increasing. Consider bidder 1 and suppose that all bidders $i > 1$ are following the strategy β_i. We show that player 1 does not have a profitable deviation.

Consider first the case in which following β_1 player 1 wins when active players are A and signals are $\mathbf{s}$: this can only happen if players in $A\setminus\{1\}$ drop at the same price, say p^*. We will show that he earns a profit, no deviations are profitable: quitting before earns him 0, and he can never change the price he pays. Without loss of generality, let $A = \{2, 3, \ldots, a\}$. Since all strategies β are increasing, all bidders in A can infer the signals $\mathbf{s}^{N\setminus A}$ of inactive bidders from the prices at which they dropped. Also, since player $i = 2, \ldots, a$ drop at p^* and

$$\beta_i^{\mathbf{s}^{N\setminus A}}(s_i) = \min\big\{p\colon \sigma^{\mathbf{s}^{N\setminus A}}(p) \geqslant s_i\big\} = p^*$$

we obtain $s_i = \sigma_i^{\mathbf{s}^{N\setminus A}}(p^*)$. Moreover, $s_1 > \sigma_1^{\mathbf{s}^{N\setminus A}}(p^*)$ and therefore $V_1^{\mathbf{s}^{N\setminus A}}(\sigma^{\mathbf{s}^{N\setminus A}}(p^*)) = p^*$ implies

$$v_1(\mathbf{s}) = v_1\big(s_1, \sigma_{-1}^{\mathbf{s}^{N\setminus A}}(p^*), \mathbf{s}^{N\setminus A}\big) > v_1\big(\sigma^{\mathbf{s}^{N\setminus A}}(p^*), \mathbf{s}^{N\setminus A}\big) = V_1^{\mathbf{s}^{N\setminus A}}\big(\sigma^{\mathbf{s}^{N\setminus A}}(p^*)\big) = p^*$$

which means that player 1 makes a profit, as was to be shown.

Consider the case in which β_1 calls for bidder 1 to drop at some price p_1^* in some sub-auction with active bidders $A = \{1, 2, \ldots, a\}$, when the other players quit at signals $\mathbf{s}^{N\setminus A}$, and suppose that bidder 1 evaluates staying longer until he wins the object. Suppose he stays until winning and that bidders quit in the order $a, a-1, a-2, \ldots, 2$ at prices $p_a \leqslant \cdots \leqslant p_2$, so that 1 wins at a price p_2. We will show that by doing this he cannot make a profit.

For p_2, the price at which player 2 quits, $s_2 = \sigma_2^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2)$ so (iii) of Lemma A.1 implies that

$$p_2 \geqslant v_1\big(\sigma_1^{\mathbf{s}^{N\setminus\{1,2\}}}(p_2), \mathbf{s}_{-1}\big). \tag{A.8}$$

Then, since for each fixed pair $(B, \mathbf{s}^{N\backslash B})$ the function $\sigma^{\mathbf{s}^{N\backslash B}}$ is increasing and when a bidder $j \in B$ drops out at p_j, we get $\sigma^{\mathbf{s}^{N\backslash B}}(p_j) = \sigma^{\mathbf{s}^{N\backslash\{B\backslash\{j\}\}}}(p_j)$ (by (iv) of Lemma A.3), we obtain

$$\begin{aligned}\sigma_1^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2) &\geqslant \sigma_1^{\mathbf{s}^{N\backslash\{1,2\}}}(p_3) = \sigma_1^{\mathbf{s}^{N\backslash\{1,2,3\}}}(p_3) \geqslant \sigma_1^{\mathbf{s}^{N\backslash\{1,2,3\}}}(p_4) = \sigma_1^{\mathbf{s}^{N\backslash\{1,2,3,4\}}}(p_4) \geqslant \cdots \\ &\geqslant \sigma^{\mathbf{s}^{N\backslash\{A\backslash\{a\}\}}}(p_a) = \sigma_1^{\mathbf{s}^{N\backslash A}}(p_a) \geqslant \sigma_1^{\mathbf{s}^{N\backslash A}}\left(p_1^*\right) = s_1 \end{aligned} \tag{A.9}$$

(the last equality follows from the fact that player 1 was supposed to quit at p_1^*). Eqs. (A.8) and (A.9) imply that $p_2 \geqslant v_1(\mathbf{s})$ so that player 1 cannot make a profit by staying longer than what his strategy calls for.

We have already shown that it is not profitable to quit when β_1 calls for staying, and it is not profitable to stay when β_1 calls for quitting. We will now show that if in some off equilibrium path, player 1 is still active at price p when he should have quit at price $p_1^* < p$, then quitting is a best response (in particular, it is better than winning at p). Let the set of active bidders at p be $J = \{1, \ldots, j\}$. Then, as in Eq. (A.9),

$$\sigma_1^{\mathbf{s}^{N\backslash J}}(p) \geqslant \sigma_1^{\mathbf{s}^{N\backslash J}}(p_{j+1}) = \sigma_1^{\mathbf{s}^{N\backslash\{J\cup\{j+1\}\}}}(p_{j+1}) \geqslant \cdots \geqslant \sigma_1^{\mathbf{s}^{N\backslash A}}(p_a) \geqslant \sigma_1^{\mathbf{s}^{N\backslash A}}\left(p_1^*\right) = s_1$$

so that $p \geqslant v_1(\sigma^{\mathbf{s}^{N\backslash J}}(p), \mathbf{s}^{N\backslash J})$ implies $p \geqslant v_1(s_1, \sigma_{-1}^{\mathbf{s}^{N\backslash J}}(p), \mathbf{s}^{N\backslash J})$. This means quitting, as his strategy prescribes, is optimal. This completes the proof that the profile of strategies defined by σ is an ex-post equilibrium. □

Proof of Theorem 3. Efficiency. Without loss of generality, suppose that at a profile of signals $\mathbf{s}$ the winner is player 1 and that the last to quit is player 2 at price p_2. Then, we have that $s_1 > \sigma_1^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2)$, $s_2 = \sigma_2^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2)$ and $v_1(\sigma^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2), \mathbf{s}^{N\backslash\{1,2\}}) = v_2(\sigma^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2), \mathbf{s}^{N\backslash\{1,2\}})$. The OEP then tells us that for

$$P = \left\{i\colon\ s_i > \left(\sigma^{\mathbf{s}^{N\backslash\{1,2\}}}(p_2), \mathbf{s}^{N\backslash\{1,2\}}\right)_i\right\}$$

we must have

$$v_1(\mathbf{s}) = \max_{i\in P} v_i(\mathbf{s}) \geqslant \max_{j\notin P} v_j(\mathbf{s})$$

establishing efficiency. □

Before proceeding to the proof of Theorem 5, we prove Lemma 8, which in turn uses this simple result.

Lemma A.5. *If v satisfies the ACC, then for all $P \subset N$ such that $j \in P$ we have that for any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$*

$$\sum_{k\in P} \frac{\partial v_k(\mathbf{s})}{\partial s_j} > |P| \frac{\partial v_i(\mathbf{s})}{\partial s_j}.$$

Proof. The proof proceeds by induction on the size of P. We already know that the result is true for $P = N$, so assume it is true for all P' with $|P'| = m + 1$. In order to obtain a contradiction, suppose that for some P with $|P| = m$, $j \in P$, and some $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and $i \neq j$ we had $\sum_P \frac{\partial v_k}{\partial s_j} \leqslant |P| \frac{\partial v_i}{\partial s_j}$. In such a case, we must have $i \in P$, since otherwise, for $P' = P \cup \{i\}$ we would have $\sum_{P'} \frac{\partial v_k}{\partial s_j} \leqslant |P'| \frac{\partial v_i}{\partial s_j}$, contradicting the induction hypothesis. We must also have

$\partial v_i/\partial s_j > \partial v_h/\partial s_j$ for all $h \notin P$, since otherwise, for some $h \notin P$ with $\partial v_i/\partial s_j \leqslant \partial v_h/\partial s_j$ we would have that for $P' = P \cup \{h\}$

$$\sum_{k\in P} \frac{\partial v_k}{\partial s_j} \leqslant |P| \frac{\partial v_i}{\partial s_j} \leqslant |P| \frac{\partial v_h}{\partial s_j} \quad \Rightarrow \quad \sum_{P'} \frac{\partial v_k}{\partial s_j} \leqslant |P'| \frac{\partial v_h}{\partial s_j}$$

contradicting the induction hypothesis. But then $\partial v_i/\partial s_j > \partial v_h/\partial s_j$ for all $h \notin P$, implies that

$$\sum_{k\in P} \frac{\partial v_k}{\partial s_j} \leqslant |P| \frac{\partial v_i}{\partial s_j} \quad \Rightarrow \quad \sum_{k\in P} \frac{\partial v_k}{\partial s_j} + \sum_{h\notin P} \frac{\partial v_h}{\partial s_j} < |P| \frac{\partial v_i}{\partial s_j} + |N \backslash P| \frac{\partial v_i}{\partial s_j}$$

$$\Leftrightarrow \quad \sum_{k\in N} \frac{\partial v_k}{\partial s_j} < |N| \frac{\partial v_i}{\partial s_j}$$

which contradicts the ACC. This concludes the proof. □

Definition 7. The set of functions v satisfies the *Equal Increments Condition* if for all $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and any $P \subset N$ there exists $j \in P$ such that for any $i \notin P$, $I_P \nabla v_j > I_P \nabla v_i$.

We present a simple lemma that will help us show that both the Average Crossing Condition and the Cyclical Crossing Condition imply the OEP. The key to showing that these conditions imply the OEP is making the connection between the effect of one signal on all valuations (as stated in the ACC and CCC) and the effect of several signals on the valuations of two players.

Lemma 8. *If v satisfies the ACC or the CCC then it satisfies the Equal Increments Condition.*

Lemma 8 asserts that when one increases the signals of a set of winners (by the same small amount) then the total growth of the valuation of one of the players whose signal increased is larger than the growth of any of those whose signals did not increase.

Proof. We first show that ACC implies the Equal Increments Condition. For all $h \in P$, and $i \notin P$, by Lemma A.5, $\sum_P \frac{\partial v_k(\mathbf{s})}{\partial s_h} > |P| \frac{\partial v_i(\mathbf{s})}{\partial s_h}$. Keeping i fixed, and adding over all $h \in P$, we obtain

$$\sum_{h\in P} \sum_{k\in P} \frac{\partial v_k(\mathbf{s})}{\partial s_h} > \sum_{h\in P} |P| \frac{\partial v_i(\mathbf{s})}{\partial s_h}.$$

We can write the previous equation as

$$\sum_{k\in P} I_P \nabla v_k > |P| I_P \nabla v_i.$$

This implies that for some $j \in P$, $I_P \nabla v_j > I_P \nabla v_i$ as was to be shown.

Now assume that v satisfies the CCC, and pick any $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and any $i \notin P$. We must show that there exists $j \in P$ such that $I_P \nabla v_j > I_P \nabla v_i$. Suppose first that there is some $k \in P$ with $k < i$ and let j be the largest k in P which is still smaller than i. In order to show that $I_P \nabla v_j > I_P \nabla v_i$, it will suffice to show that for all $k \in P \backslash \{j\}$, $\partial v_j/\partial s_k \geqslant \partial v_i/\partial s_k$, since then $\partial v_j/\partial s_j > \partial v_i/\partial s_j$ will make the desired inequality strict. Notice that for all $k < i$, we have $k < j < i$, so by the CCC, we have $\partial v_j/\partial s_k \geqslant \partial v_i/\partial s_k$. For $k > i$, we have that the CCC tells us that

$$\frac{\partial v_k}{\partial s_k} > \frac{\partial v_1}{\partial s_k} \geqslant \frac{\partial v_j}{\partial s_k} \geqslant \frac{\partial v_i}{\partial s_k}.$$

Suppose now that for the chosen i there is no $k < i$ in P. For $j \equiv \max_{k \in P} P$ we will show, as before, that for all $k \in P \backslash \{j\}$, $\partial v_j / \partial s_k \geqslant \partial v_i / \partial s_k$. Notice that for all $k \in P \backslash \{j\}$ we have $i < k < j$, so the CCC tells us that

$$\frac{\partial v_k}{\partial s_k} > \frac{\partial v_j}{\partial s_k} \geqslant \frac{\partial v_n}{\partial s_k} \geqslant \frac{\partial v_1}{\partial s_k} \geqslant \frac{\partial v_i}{\partial s_k}$$

as was to be shown. □

Proof of Theorem 5. Suppose the OEP is violated, so that there exists an $\mathbf{s}$ with $|W(\mathbf{s})| > 1$ and an $\mathbf{s}' \geqslant \mathbf{s}$ such that $s'_j > s_j$ if and only if $j \in P \subset W(\mathbf{s})$, and that for all $j \in P$, $v_j(\mathbf{s}') < v_i(\mathbf{s}')$ for some i. Without loss of generality, suppose $P = \{1, \ldots, m\}$ and assume also without loss of generality, that $s'_1 - s_1 \leqslant s'_2 - s_2 \leqslant \cdots \leqslant s'_m - s_m$. Define

$$\alpha_1 = \max\bigl\{\alpha\colon \exists j \in P,\ v_j(\mathbf{s} + I_P \alpha) \geqslant v_i(\mathbf{s} + I_P \alpha)\ \forall i \notin P\bigr\}.$$

Note that if for any $\alpha \leqslant s'_1 - s_1$ we had that for some $i \notin P$, and $\forall j \in P$, $v_j(\mathbf{s} + I_P\alpha) < v_i(\mathbf{s} + I_P\alpha)$, there would be some $\alpha^* \in [0, \alpha)$ such that for some $j \in P$ and $i \notin P$: $v_j(\mathbf{s} + I_P\alpha^*) = v_i(\mathbf{s} + I_P\alpha^*)$ and for all $\varepsilon > 0$, $v_k(\mathbf{s} + I_P[\alpha^* + \varepsilon]) < v_i(\mathbf{s} + I_P[\alpha^* + \varepsilon])$ for all $k \in P$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = \mathbf{s} + I_P\alpha^*$ we have $|W(\mathbf{s}')| > 1$, and that $I_P \nabla v_k(\mathbf{s} + I_P\alpha^*) \leqslant I_P \nabla v_i(\mathbf{s} + I_P\alpha^*)$ for all $k \in P$, which contradicts the Equal Increments Condition, and would thus by Lemma 8 conclude the proof. Assume then $\alpha_1 > s'_1 - s_1$.

Define then $P_2 = P \backslash \{1\}$, and

$$\alpha_2 = \max\bigl\{\alpha\colon \exists j \in P_2,\ v_j\bigl(\bigl(s'_1, \mathbf{s}_{-1}\bigr) + I_{P_2}\alpha\bigr) \geqslant v_i\bigl(\bigl(s'_1, \mathbf{s}_{-1}\bigr) + I_{P_2}\alpha\bigr)\ \forall i \notin P_2\bigr\}.$$

Since for all $\alpha \leqslant s'_1 - s_1$, $\exists j \in P$, such that $v_j(\mathbf{s} + I_P\alpha) \geqslant v_i(\mathbf{s} + I_P\alpha)$ $\forall i \notin P$, we have that $\alpha = s'_1 - s_1$ belongs to the set over which the max is taken, so α_2 is well defined. If we had $\alpha_2 \leqslant s'_2 - s_2$, we would obtain that there are $j \in P_2$ and $i \notin P_2$ such that $v_j((s'_1, \mathbf{s}_{-1}) + I_{P_2}\alpha_2) = v_i((s'_1, \mathbf{s}_{-1}) + I_{P_2}\alpha_2)$ and that for all $\varepsilon > 0$, $v_k((s'_1, \mathbf{s}_{-1}) + I_{P_2}[\alpha_2 + \varepsilon]) < v_i((s'_1, \mathbf{s}_{-1}) + I_{P_2}[\alpha_2 + \varepsilon])$ for all $k \in P_2$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = (s'_1, \mathbf{s}_{-1}) + I_{P_2}\alpha_2$ we have $|W(\mathbf{s}')| > 1$, and that for all $k \in P_2$

$$I_{P_2} \nabla v_k(\mathbf{s}') \leqslant I_{P_2} \nabla v_i(\mathbf{s}')$$

which contradicts the Equal Increments Condition, and would thus by Lemma 8 conclude the proof.

Fix some $l \leqslant m$ and define $\widetilde{\mathbf{s}} = (s'_1, \ldots, s'_{l-1}, s_l, s_{l+1}, \ldots, s_n)$ and $P_l = P \backslash \{1, \ldots, l-1\}$. As an induction hypothesis, suppose that for some $j \in P_l$, $v_j(\widetilde{\mathbf{s}}) \geqslant v_i(\widetilde{\mathbf{s}})$ for all $i \notin P_l$ (we have already proved this for $l = 1$ and $l = 2$) and define

$$\alpha_l = \max\bigl\{\alpha\colon \exists j \in P_l,\ v_j(\widetilde{\mathbf{s}} + I_{P_l}\alpha) \geqslant v_i(\widetilde{\mathbf{s}} + I_{P_l}\alpha)\ \forall i \notin P_l\bigr\}.$$

Again, if we had $\alpha_l \leqslant s'_l - s_l$ we would obtain that there are $j \in P_l$ and $i \notin P_l$ such that $v_j(\widetilde{\mathbf{s}} + I_{P_l}\alpha_l) = v_i(\widetilde{\mathbf{s}} + I_{P_l}\alpha_l)$ and that for all $\varepsilon > 0$, $v_k(\widetilde{\mathbf{s}} + I_{P_l}[\alpha_l + \varepsilon]) < v_i(\widetilde{\mathbf{s}} + I_{P_l}[\alpha_l + \varepsilon])$ for all $k \in P_l$. Taking derivatives with respect to ε and evaluating at $\varepsilon = 0$, we obtain that for $\mathbf{s}' = \widetilde{\mathbf{s}} + I_{P_l}\alpha_l$ we have $|W(\mathbf{s}')| > 1$, and that $I_{P_l} \nabla v_k(\mathbf{s}') \leqslant I_{P_l} \nabla v_i(\mathbf{s}')$ for all $k \in P_l$. This contradiction concludes the proof. □

Proof of Theorem 4. Suppose there is an interior $\mathbf{s}$, and an $\mathbf{s}' \geqslant \mathbf{s}$ with $s'_j > s_j$ iff $j \in P \subset W(\mathbf{s})$ but that $\max_{j\colon s'_j > s_j} v_j(\mathbf{s}') < \max_{k\colon s'_k = s_k} v_k(\mathbf{s}')$ (i.e. $W(\mathbf{s}') \cap P = \emptyset$). Suppose, without loss of generality that $P = \{1, 2, \dots, m\}$ and $W(\mathbf{s}) = \{1, 2, \dots, k\}$ for $k \geqslant m$. For all $i \notin W(\mathbf{s})$ and $j \in W(\mathbf{s})$ we have $v_i(\mathbf{s}) < v_j(\mathbf{s})$. By continuity of the v functions, there is $\varepsilon_1 > 0$ such that for all

$$\left(s_1^*, \dots, s_k^*\right) \in B_{\varepsilon_1}^k(\mathbf{s}) = \left\{s^* \in \mathbf{R}^k\colon\ \left\|s^* - (s_1, \dots, s_k)\right\| < \varepsilon_1\right\}$$

we have $v_i(s_1^*, \dots, s_k^*, s_{k+1}, \dots, s_n) < v_j(s_1^*, \dots, s_k^*, s_{k+1}, \dots, s_n)$ for all $i \notin W(\mathbf{s})$ and $j \in W(\mathbf{s})$. Moreover, since $W(\mathbf{s}') \cap P = \emptyset$, there exists a small $\varepsilon_2 > 0$ such that for all

$$\left(s_1^*, \dots, s_k^*\right) \in B_{\varepsilon_2}^k(\mathbf{s}) = \left\{s^* \in \mathbf{R}^k\colon\ \left\|s^* - (s_1, \dots, s_k)\right\| < \varepsilon_2\right\}$$

we also have that for $\widetilde{\mathbf{s}} = (s_1', \dots, s_m', s_{m+1}^*, \dots, s_k^*, s_{k+1}, \dots, s_n)$, $W(\widetilde{\mathbf{s}}) \cap P = \emptyset$ (as $W(\mathbf{s}') \cap P = \emptyset$).

Then, define $B = B_{\varepsilon_1}^k(\mathbf{s}) \cap B_{\varepsilon_2}^k(\mathbf{s})$, and for each $j \in W(\mathbf{s})$, define $v_j^k\colon B \to \mathbf{R}$ by $v_j^k(\mathbf{x}) = v_j(\mathbf{x}, s_{k+1}, \dots, s_n)$. Since $\mathbf{s}$ is interior and (by regularity) the Jacobian of $v^k = (v_1^k, \dots, v_k^k)$ is invertible at $(s_1, s_2, \dots, s_k)$, the Inverse Function Theorem ensures that one can find $(s_1^*, \dots, s_m^*, \dots, s_k^*) \in B$ such that:

(a) for $\mathbf{s}^* = (s_1^*, \dots, s_k^*, s_{k+1}, \dots, s_n)$, $W(\mathbf{s}^*) = P$ (by the Inverse Function Theorem, we can reduce the valuation v_i for all i in $W(\mathbf{s})\backslash P$, while keeping those of players in P constant; for players i not in $W(\mathbf{s})$ the fact that $(s_1^*, \dots, s_m^*, \dots, s_k^*) \in B \subset B_{\varepsilon_1}^k(\mathbf{s})$ ensures that $v_i(s_1^*, \dots, s_k^*, s_{k+1}, \dots, s_n) < v_j(s_1^*, \dots, s_k^*, s_{k+1}, \dots, s_n)$ for any $j \in W(\mathbf{s})$);

(b) for $\widetilde{\mathbf{s}} = (s_1', \dots, s_m', s_{m+1}^*, \dots, s_k^*, s_{k+1}, \dots, s_n)$, $W(\widetilde{\mathbf{s}}) \cap P = \emptyset$.

Then, let $\mu(\mathbf{s}^*) = \mu(\widetilde{\mathbf{s}}) = 1/2$. Since players not in P have no uncertainty in their type, their bidding functions are just a (possibly) mixed strategy independent of the type. Let $\overline{\beta}$ be the maximum element in the union of the supports of all bidding functions of players not in P (when all players in P are active). Also, for players in P it is a dominant strategy to bid their valuations (that have no uncertainty), so let $\beta_i^* = v_i(\mathbf{s}^*)$ and $\widetilde{\beta}_i = v_i(\widetilde{\mathbf{s}})$ for all $i \in P$ (note that since $P = W(\mathbf{s}^*)$ by item (a), $\beta_i^* = \beta_j^*$ for all $i, j \in P$).

By efficiency, and $W(\mathbf{s}^*) = P$ (item (a)), we must have that all players not in P must quit with probability 1 before the price reaches β_i^*: for all $i \in P$,

$$\beta_i^* \geqslant \overline{\beta}. \tag{A.10}$$

But since $W(\widetilde{\mathbf{s}}) \cap P = \emptyset$ (item (b)) efficiency implies that we must also have that

$$\overline{\beta} > \max_{i \in P} \widetilde{\beta}_i = \max_{i \in P} v_i(\widetilde{\mathbf{s}}). \tag{A.11}$$

Then, since v_i is increasing at ties and strictly increasing in s_i, we have that for all $i \in P$, $v_i(\widetilde{\mathbf{s}}) > v_i(\mathbf{s}^*)$. This implies, together with Eqs. (A.10) and (A.11), that for all $i \in P$,

$$\beta_i^* \geqslant \overline{\beta} > \max_{i \in P} \widetilde{\beta}_i = \max_{i \in P} v_i(\widetilde{\mathbf{s}}) > \max_{i \in P} v_i(\mathbf{s}^*) = \max_{i \in P} \beta_i^*$$

which is a contradiction. □

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