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A correction to uniqueness in "Competitive Bidding and Proprietary Information"

Juan Dubra

Universidad de Montevideo, Prudencio de Pena 2440, Montevideo 11600, Uruguay

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Abstract

In this note I show that there is a mistake in the proof of uniqueness in Engelbrecht-Wiggans, Milgrom and Weber's seminal "Competitive Bidding and Proprietary Information" and provide a correct proof.

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1. Introduction

One of the best known models of auction theory involves an informed bidder competing for a common value object against one or more uninformed bidders. There are at least three reasons why this model became so well known. First, it applies to a wide variety of situations of interest. Second, the model and its variations perform well when matched with the data, as has been shown by Hendricks and Porter (1988) and Hendricks et al. (1994) among others. Finally, its solution is simple, intuitive and elegant.

This type of auction was described by Woods (1965) and first studied formally by Wilson (1967) who found an equilibrium of the bidding game. The formal model was later studied by Weverbergh (1979) who found a mistake in Wilson's existence proof, and used the same (restrictive) assumptions as Wilson to find an equilibrium. Hughart (1975) found an equilibrium of the same game under a different set of assumptions. Finally, Engelbrecht-Wiggans et al. (1983)

E-mail address: dubraj@um.edu.uy.

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(EMW for short) found an equilibrium to the game using much weaker assumptions. The theoretical results arising from this model have been extended to a variety of set-ups by, among others, Milgrom and Weber (1982) and Hendricks et al. (1994).

In general, in order to test empirically the predictions of any given model, one needs uniqueness of equilibrium. Therefore, part of the reason why the model I analyze here became so well-known, is that EMW claimed to have proved uniqueness. In this note I show that there is a mistake in their proof of uniqueness and provide a correct proof.

2. The model

Player 1, the informed party, observes (h, u) where h is drawn from any distribution F with bounded support and u is drawn independently from an atomless distribution. Players $2, \ldots, N$ make no observations. The value of the object for all players is h, and the object is sold using a first price auction. Although this is not explicit in EMW, the strategy space S_i for the uninformed player i is the space of distributions in $\mathbf{R}_+$:

$$S_i = \{G_i : G_i \text{ is a distribution on } \mathbf{R}_+\}$$

Let

$$\beta(h, u) = E(H|H < h \text{ or } (H = h \text{ and } U < u))$$

denote a strategy for the informed player. As EMW (correctly) argue, this is the unique equilibrium bidding strategy of player 1.[1] Furthermore, this uniqueness is established *without* assuming that player 2's equilibrium strategy is unique.

EMW goes on to claim that, for $G = G_2 \cdots G_N$:

Theorem 1. *The N-tuple* $(\beta, G_2, G_3 \ldots, G_N)$ *is an equilibrium only if*

$$G(b) = P(\beta(h, u) \leq b) \tag{1}$$

3. The problem

EMW's proof of the claim proceeds by asserting that since β is optimal, $\beta(h, u)$ solves

$$\max_b (h - b)G(b)$$

with first order necessary condition

$$(h - b)G'(b) = G(b), \tag{2}$$

and that since this is a first order linear differential equation in G, on a convex domain, with the terminal condition $G(E(h)) = 1$, the solution is unique.

Note that the assertion that Eq. (2) is a differential equation, requires that the equilibrium G be differentiable (*everywhere*, not merely almost everywhere). However, as I now show, the equilibrium proposed by the authors themselves in Eq. (1) may not be differentiable.

[1] This strategy is essentially unique, in the sense that one can re-order the noise variable u, and obtain another equilibrium strategy. I thank a referee for pointing this out.

Example 1. Let $f : [0, 2] \to \mathbf{R}$ be a density defined by

$$f(h) = \begin{cases} \dfrac{1}{4} & h \in [0, 1] \\ \dfrac{3}{4} & h \in (1, 2] \end{cases}$$

so that the distribution is

$$F(h) = \begin{cases} \dfrac{h}{4} & h \in [0, 1] \\ \dfrac{3}{4}h - \dfrac{1}{2} & h \in (1, 2] \end{cases}$$

$$\beta(h) = E(H|H < h) = \begin{cases} \dfrac{h}{2} & h \in [0, 1] \\ \dfrac{1}{2}\dfrac{3h^2 - 2}{3h - 2} & h \in (1, 2] \end{cases}$$

Then, the probability that the uninformed bidder bids less than b is the probability that $\beta(h)$ is less than b:

$$G(b) = P(\beta(h) < b) = \begin{cases} \dfrac{b}{2} & b \in \left[0, \dfrac{1}{2}\right] \\ \dfrac{3}{4}b + \dfrac{1}{4}\sqrt{9b^2 + 6 - 12b} - \dfrac{1}{2} & b \in \left(\dfrac{1}{2}, \dfrac{5}{4}\right] \end{cases}$$

which is not differentiable:

$$\left.\frac{d((3/4)b + (1/4)\sqrt{9b^2 + 6 - 12b} - (1/2))}{\mathrm{d}b}\right|_{b=1/2} = \frac{3}{2}$$

In the next section, I present an alternate proof of uniqueness.

4. A proof of uniqueness

If F is degenerate, the problem is trivial, so assume F is non-degenerate. Suppose $(\beta, G_2, G_3 \ldots, G_N)$ and $(\beta, J_2, J_3 \ldots, J_N)$ are two equilibria (recall that we already know that player 1 has only one equilibrium strategy). Since player 1 never bids above $E(h)$, clearly

(a) $G(E(h)) = J(E(h)) = 1$.
I now show that

(b) G_i and J_i are continuous for all i. Suppose not and suppose that, say, G_i is not continuous, so that for some $B \in [0, E(h)]$ and some $j > 0$, for all $\varepsilon > 0$, $G(B) - G(B - \varepsilon) \geq j$. Then for some $\delta > 0$ all types h whose bid distributions have support intersecting $(B - \delta, B)$, are strictly better off bidding slightly above B, contradicting the fact that the support of 1's equilibrium bids is $[0, E(h)]$.
Let $\underline{h}$ be the minimal element of the support of F and $\overline{h}$ its maximal element. If the two equilibria are different, there must be some $b_{\neq} \in [0, E(h))$ such that $G(b_{\neq}) \neq J(b_{\neq})$, so

suppose without loss of generality that $G(b_{\neq}) > J(b_{\neq})$ and define $K = G - J$. Since by (b) K is continuous, let $[b_{\neq}, b_{=}]$ be the unique interval on which

(c) $K(b) > 0$ for all $b \in [b_{\neq}, b_{=})$ and $K(b_{=}) = 0$.

Let $h_{\neq}$ be a type for which $b_{\neq}$ is optimal, and let $h_{=}$ be a type for which $b_{=}$ is optimal. I now show that $h_{\neq} \neq h_{=}$. Suppose to the contrary that $h_{\neq} = h_{=}$. Since $b_{\neq} < b_{=}$, we have that $h_{\neq} > \underline{h}$ because the only equilibirum bid of $\underline{h}$ is $\underline{h}$. If type $h = h_{\neq} = h_{=}$ makes two different equilibrium bids, there must be an atom at h. Therefore, using (b) there is an interval $I = (b_{\neq}, b_{\neq} + \varepsilon)$ such that for all $b \in I$, $G(b) > J(b)$ and every b in the interval is an equilibrium bid of $h_{\neq}$. Since $b_{=}$ is an equilibrium bid of $h_{\neq}$, almost every equilibrium bid of $h_{\neq}$ must yield the same payoff as bidding $b_{=}$, namely $(h_{\neq} - b_{=})G(b_{=})$ which, by (c), equals $(h_{\neq} - b_{=})J(b_{=})$. Therefore, for almost every $b \in I$, we have

$$(h_{\neq} - b)G(b) = (h_{\neq} - b)J(b).$$

Because F is non degenerate and $h_{\neq} > \underline{h}$, $h_{\neq} > b$, which together with the last equation yield $G(b) = J(b)$, a contradiction.

Let $\mathcal{H}$ denote the convex hull of the support of F, and let b: $\mathcal{H} \to \mathbf{R}$ be any selection from $\beta(\cdot, u)$ such that $b(h_{\neq}) = b_{\neq}$ and $b(h_{=}) = b_{=}$ (note that β is well defined for all $h \in \mathcal{H}$). For all types $h \in (h_{\neq}, h_{=})$, if b is an equilibrium bid of h, then $b \in (b_{\neq}, b_{=})$, and so $K(b) > 0$. That is,

(d) for all $h \in (h_{\neq}, h_{=})$, $K(b(h)) > 0$

By Theorem 2 of Milgrom and Segal (2002), for all $h \in [h_{\neq}, h_{=}]$,

$$\begin{aligned}(h - b(h))G(b(h)) &= (\underline{h} - b(\underline{h}))G(b(\underline{h})) + \int_{\underline{h}}^{h} G(b(s))\,\mathrm{d}s \\ (h - b(h))J(b(h)) &= (\underline{h} - b(\underline{h}))J(b(\underline{h})) + \int_{\underline{h}}^{h} J(b(s))\,\mathrm{d}s\end{aligned} \tag{3}$$

so that

$$\begin{aligned}(h - b(h))K(b(h)) &= (h - b(h))[G(b(h)) - J(b(h))] \\ &= (\underline{h} - b(\underline{h}))K(b(\underline{h})) + \int_{\underline{h}}^{h} K(b(s))\,\mathrm{d}s\end{aligned} \tag{4}$$

The following inequalities constitute a contradiction, proving that $G = J$:

$$\begin{aligned}0 &= (h_{=} - b(h_{=}))K(b(h_{=}))(\text{definition of } h_{=}) = (\underline{h} - b(\underline{h}))K(b(\underline{h})) \\ &\quad + \int_{\underline{h}}^{h_{=}} K(b(s))\,\mathrm{d}s\,(\text{Eq. (4)}) = (\underline{h} - b(\underline{h}))K(b(\underline{h})) \\ &\quad + \int_{\underline{h}}^{h_{\neq}} K(b(s))\,\mathrm{d}s + \int_{h_{\neq}}^{h_{=}} K(b(s))\,\mathrm{d}s \geq (\underline{h} - b(\underline{h}))K(b(\underline{h})) \\ &\quad + \int_{\underline{h}}^{h_{\neq}} K(b(s))\,\mathrm{d}s(\text{literal(d)}) = (h_{\neq} - b(h_{\neq}))K(b(h_{\neq})) \\ &\quad > 0(h_{\neq} > b(h_{\neq}) \text{ and } K(b(h_{\neq})) > 0)\end{aligned}$$

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References

Engelbrecht-Wiggans, Milgrom, R.P., Weber, R., 1983. Competitive bidding and proprietary information. Journal of Mathematical Economics 11 (2), 161–69.

Hendricks, K., Porter, R., 1988. An empirical study of an auction with asymmetric information. American Economic Review 78 (5), 865–883.

Hendricks, K., Porter, R., Wilson, C., 1994. Auctions for oil and gas leases with an informed bidder and a random reservation Price. Econometrica 63 (1), 1–27.

Milgrom, P., Weber, R.J., 1982. The value of information in a sealed-bid auction. Journal of Mathematical Economics 10 (1), 105–114.

Milgrom, P., Segal, I., 2002. Envelope theorems for arbitrary choice sets. Econometrica 70 (2), 583–601.

Weverbergh, M., 1979. Competitive bidding with asymmetric information reanalyzed. Management Science 25, 291–294.

Wilson, R., 1967. Competitive bidding with asymmetric information. Management Science 13, A816–A820.

Woods, D.H., 1965. Decision Making under Uncertainty in Heirarchial Organizations, dissertation, Harvard Business School, Boston.