Abstract

Management compensation packages typically include some form of bonus payment; these payments shall provide incentives for effort, but recently they were heavily criticized for their perceived excessive risk-taking incentives. This paper presents a continuous-time model in which the manager decides on the risk and effort level of the bank. A single-period model is nested into continuous-time; for this we confirm prior results on risk and effort levels and show that they misrepresent actual risks taken in our dynamic continuous-time setup. Furthermore, we show that with current bonus schemes risk and effort levels are typically of bang-bang type: the manager takes either maximal or minimal risk and either maximal or minimal effort. The bonus-malus scheme is believed by the public to limit risk-taking; we show that before the deferral date it behaves like a call option type bonus scheme and thereafter considerably hampers risk and effort. We also discuss how a bonus scheme should look like that incentivizes the manager to pick a target risk level. Finally, we illustrate our results quantitatively.

Keywords

bonus, risk, effort, incentives, options
1 Introduction

The public, the government and regulatory bodies attribute the current financial crisis to a culture for excessive risk taking, see the Financial Stability Board (2009) and the G-20 Leader’s Statement (2009).1 Bonus schemes in banks have come under heavy scrutiny for their asymmetric relationship: managers seem to reap huge benefits if business is doing well but do not share in losses if it turns sour. From a finance perspective, bonus schemes are intended to incentivize managers to provide effort. This paper studies in a dynamic continuous-time model what effort incentives bonus schemes give and what risk taking incentives they imply.

We study a continuous-time model in which the asset value follows a geometric Brownian motion. The manager controls volatility and can increase the drift through effort beyond fair pricing. While effort increases the current price of assets it comes at a disutility to the manager; the manager therefore receives a bonus payment that depends on the asset value. We determine the fair value of the manager’s claim using the risk-neutral pricing measure, see, e.g., Duffie (2001), and assume the manager decides on the risk and effort levels that will maximize the fair value of her personal claim. The paper analyzes various bonus schemes both from a static and a dynamic perspective.

Our valuation setup is a standard risk-neutral pricing framework, because in principle the asset values in our model come from securities that are traded on exchanges. We could model the dynamics of different securities to then derive from these the portfolio asset dynamics; however this would lead us away from our focus on risk and effort incentives for the manager. Because the only source of risk in our model is a single Wiener process we assume the market provides this market price of risk and in such a way that only the systematic risk component is priced.

Modeling manager’s preferences would permit us to determine the individual value they attribute to their bonus claim. Power utility is the most common type of preferences used in finance; in our geometric Brownian motion setup, Rubinstein (1976) and Brennan (1979) showed

1A culture of excessive risk taking is acutely characterized by Lewis (1990) in his book “Liars Poker;” he records the words of a senior trader at Salomon Brothers “At other places, management says, ‘Well, gee, fellas, do we really want to bet the ranch on this deal?’” and reports his attitude: “Sure, what the fuck, its only a ranch.”

2Our valuation approach is closely related to the real-options approach, see, e.g. Dixit and Pindyck (1996) and Trigeorgis (1996); but from a conceptual viewpoint within the real options literature the underlying asset value is not traded, while we believe that in our setup it is traded.
that power utility will lead to risk-neutral pricing of the asset value. We believe managers are well aware of market movements and able to determine the fair value of their personal claims. Cadenillas, Cvitanic, and Zapatero (2004), and Duan and Wei (2005) proceeded similarly: they did not model manager’s preferences explicitly and focused instead on the fair price to study bonus schemes.

In a recent survey, the Institute of International Finance (2009) reports on p. 20 that 31% of respondents use a formal payout function to determine bonus payments. Jensen, Murphy, and Wruck (2004) provide an overview on bonus schemes and discuss extensively their respective strengths and weaknesses. Throughout this paper we consider a linear bonus scheme (called “stock bonus”), a bonus scheme that resembles a call option because it pays nothing below a threshold and increases linearly thereafter (called “simple bonus”), and a bonus scheme in which the payout from the previous simple bonus is limited (called “capped bonus”). The capped bonus is a popular scheme among many companies, see Murphy (2000). The Institute of International Finance (2009) reports in its survey that 14% of respondents use the stock bonus and 17% use a so-called S-curve bonus. (The S-curve bonus function is linear over intervals but the slope changes in such a way that the curve resembles an S-curve. We do not pursue this here because we want to focus on simple schemes and the S-curve can be treated as a composition of simple bonus schemes.) In addition we study the so-called bonus-malus scheme that Union Bank of Switzerland recently proposed and implemented; at a first so-called deferral (decision) date a fraction of the bonus is paid out; the other part is deferred and reduced if losses amount at a later date.

Comparing the static with the dynamic setup we find considerable differences: while the incentive schemes lead to maximal manager effort in the single-period analogue, in the continuous-time setup the manager will not provide effort for a large range of asset values over time. Furthermore, while risk may be kept under control with the capped bonus and the bonus-malus scheme in the single-period analogue, in the continuous-time setup the manager chooses mostly maximal risk and, if not, usually minimal risk.

In general, optimal actions in continuous-time are often of bang-bang type: the manager takes either maximal or minimal risk, and either minimal or maximal effort. We motivate this by taking the limit shrinking to zero the time to the final bonus payment. In continuous-time we
show for the bonus-malus scheme that the total claim value is similar to a simple bonus at the
deferral decision date, i.e. at the date part of the bonus is paid and the other part deferred; the
bonus-malus then leads to bang-bang solutions for risk and effort after the deferral decision date
while before the date the bonus-malus scheme provides incentives similar to the simple bonus.

Because we find current bonus scheme to be disappointing risk incentivization devices, we look
for bonus schemes that have the manager pick a target risk level; while we show it is not possible
to do so independent of the asset value, we can achieve this for all above a critical asset value.
The designed scheme is zero below the critical asset value and continuously increasing thereafter
with negative curvature. Numerical simulations show that this critical value is reasonable for
the practice and that the resulting bonus scheme resembles partially a smoothed version of the
popular capped bonus scheme.

Most of the incentive literature focuses on bonus schemes as contract design within principal-
agent theory; for example, Holmström and Milgrom (1987) show in a multiperiod setup that
optimal compensation contracts will be “linear” in an appropriate sense, while Sung (1995)
studies an extension to continuous-time Brownian motion with drift. This strand of the literature
has provided very valuable qualitative insights, but they do not contribute to our quantitative
understanding of functional forms of bonus schemes, risk and effort.

The literature on risk taking implications of incentive schemes studies the impact of diversifi-
cation and managerial risk-aversion on effort and risk, e.g. Meulbroek (2001) and Jin (2002). In
a continuous-time stochastic volatility model Duan and Wei (2005) discuss simple bonus schemes
(“call options”), their incentives to increase idiosyncratic risk not only total risk and conclude
that payments should be indexed. Cadenillas, Cvitanic, and Zapatero (2004) study a geometric
Brownian motion setup, where a manager with log utility decides on the firm’s leverage; they
find that high-type managers prefer simple bonus schemes (“call options”) while low-type man-
agers prefer stock (linear contracts). Our setup focuses on a single asset and on the impact of
idiosyncratic risk on we do not study the impact of systematic risk on pricing. This strand of
the literature made important contributions to our understanding of bonus schemes; however,
we are less concerned about diversification. Instead, differently to the prior literature, we are
interested in the dynamic behavior of risk and effort choice over time and we want to compare
this dynamic behavior for different bonus schemes.
Our paper makes the following contributions. First of all, while previous papers studied stock versus call options (“simple bonus schemes”) we study a wide array of bonus schemes, including the recent bonus-malus model and derive mostly explicit expressions about the form of the optimal risk and effort levels for all these bonus schemes both in a static single-period setup and in a dynamic continuous-time setup. Second, our paper points out considerable differences between static and dynamic models and shows that most bonus schemes lead to disappointing bang-bang solution in risk and effort over in continuous-time; in particular we show that the proposed bonus-malus model shares these poor features. Third, we discuss how bonus schemes should look like for which the manager will choose target risk levels.

The remainder of the paper is organized as follows. The next two sections introduce our model and the incentive schemes. Section 4 assumes that the manager fixes her risk and effort levels right from the start and leaves them unchanged; this corresponds to a single-period setup and allows us to compare our results with prior ones from the literature. Section 5 studies the general setup considered in our paper; it discusses what determines continuous-time optimal risk and effort levels of the manager, in general and specifically for current bonus schemes. Section 6 adopts chooses parameters to illustrate our results quantitatively. The paper concludes with section 7. All proofs are postponed to the appendix.

2 The Model

We study the directors and officers of a bank as a group, called the (bank) manager, over a finite time-horizon $[0, T]$. (Time 0 is today.) The bank starts with (balance sheet) assets in the amount 1. This assumption is without loss of generality; it merely means that all values are expressed as fractions of date 0 asset value. We assume the continuously compounded interest rate $r$ is constant and at each point in time $t$ the manager can invest into only one risky investment opportunity.

2.1 Growth of Bank Assets

The dynamics of bank’s assets is

$$dA_t = (\mu(\sigma_t) + \theta_t \sigma_t) A_t dt + \sigma_t A_t dW_t,$$

(1)
where $W$ is a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t)$, where the filtration $\mathcal{F}_t = \sigma(W_s | 0 \leq s \leq t)$ is generated by the Wiener process $W$. Here $P$ denotes the physical (objective) probability measure, $\lambda$ the market price of risk and the function

$$\mu(\sigma) = r + \lambda \sigma$$

the continuously compounded mean return ("drift rate") for any risk-level $\sigma$. We assume $\sigma = (\sigma_t)_t, \theta = (\theta_t)_t$ are adapted processes.

Note that $\sigma_t$ enters multiplicatively with the Wiener process $W$ and so we refer to this as the risk-level. The asset drift rate in our model is given through a market-based term $\mu(\sigma_t)$ and an additional term $\theta_t \sigma_t$. The market-based term takes care of the pricing for risk that prevails in the market; the additional term then plays the role Jensen’s alpha. As usual in asset management we decompose this here into the product $\theta_t \sigma_t$, where the parameter $\theta_t$ plays the role of the information ratio. The fundamental theorem of active management tells us that the information ratio depends on effort and will be zero without effort (market pricing); it scales linearly in the risk level chosen. Therefore we refer to $\theta$ as the effort-level.

The main business of banks is risk-taking, e.g. through provision of credit, proprietary trading or investment banking in general. Our model permits setting different risk-levels which could be achieved by the manager through partial investing in safe assets, diversification or leveraging. We do not think about how assets splits into debt and equity and do not permit bank manager to invest into riskless investments. Because our focus is on the value of total assets and the impact the manager’s incentives have, we do not model how different risk-levels can be achieved; we assume they can.

2.2 Desired Actions by Bank Owner and Manager

While the bank manager invests the moneys, these are ultimately provided by the bank’s owners. We assume the owners cannot enforce a risk or an effort level, e.g. because they do not have the market knowledge to see the risk-level chosen and/or because effort and risk are mingled so that it cannot be enforced in courts of justice. We assume $0 \leq \sigma_{\text{min}} < \sigma_{\text{max}}$ and $0 < \theta_{\text{max}}$. At any

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3In our model we are interested in chosen risk and effort levels; therefore do not characterize explicitly diversification and how risk splits into systematic and unsystematic components. Implicitly, reflects the appropriate beta contribution from stochastic discount factor through the market-price of risk $\lambda$. 

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point in time the manager may unilaterally decide on the risk level \( \sigma_{\text{min}} \leq \sigma_t \leq \sigma_{\text{max}} \); in addition the manager can provide effort at the level \( 0 \leq \theta_t \leq \theta_{\text{max}} \). In terms of stochastic optimization these are so-called control variables.

We assume the manager will decide the risk and effort level over time through processes which maximize the market value of her claim and denote these as \( \theta^*_M, \sigma^*_M \). The owner cannot decide on these levels; however, for comparison we derive the the risk and effort level which the owner desires; these are the levels which maximize market values of her respective claim and we denote these as \( \theta^*_O, \sigma^*_O \). Note that this is a stochastic programming problem that essentially corresponds to a classical present value maximization. Our approach on valuation will be the theme of section 2.3.

Any effort \( \theta_t \) by the manager at a time \( t \) comes at some disutility which we translate into the monetary cost \( \gamma(\theta_t) \). This cost will be paid at all times. The cost function \( \gamma \) has the property that

\[
\gamma(0) = 0, \gamma(\theta) > 0, \gamma'(\theta) > 0, \gamma''(\theta) \geq 0 \quad \text{for all } 0 \leq \theta \leq \theta_{\text{max}}.
\]

The first property states that more effort is costly; the second property states that marginal costs cannot decrease, i.e. the higher the current level of effort, the marginal cost cannot decrease. When \( \gamma'' > 0 \), each additional unit comes at a higher cost and this makes incentive contracting all the harder. If \( \gamma'' = 0 \), then each additional unit of effort has the same cost; it is a typical one in microeconomics. Throughout most of this paper we study a general cost function; only in the parameterization section 6 we restrict ourselves to a functional form of \( \gamma \) to illustrate our results.

In general we expect the bank manager to desire the maximum effort level; however, without appropriate compensation package, we expect the manager not to provide effort at any time \( (\theta = 0) \), because then costs would accrue to him, but he would not receive any benefit. The owners therefore need to provide appropriate incentives. Throughout we ignore base compensation; doing so would merely add an additional parameter to our analysis that would not be relevant for risk-taking an effort. Incentive schemes will be introduced in section 3.

\[\text{Footnote 4: This is similar in spirit to Grossman and Hart (1983).}\]
2.3 Valuing the Claims

To value claims we switch to the so-called risk-neutral pricing measure $Q$ (a.k.a. the equivalent martingale measure) with the property that the process $W_t^Q = \lambda t + W_t$ is a standard Wiener process under $Q$. Then today’s price of a claim on the asset process, that pays at some time $t$ according to a $\mathcal{F}_t$ measurable payoff function $g_t$, is

$$E \left[ e^{-rt} g_t \right].$$

(3)

Throughout this paper we only study payoffs at one or two dates. The asset dynamics becomes under $Q$ for the chosen effort and risk-levels:

$$dA_t = (r + \theta_t \sigma_t)A_t dt + \sigma_t dW_t^Q.$$  

(4)

Note that, if effort $\theta = 0$ at all times, then whatever the risk-level $\sigma$ chosen, the price of assets is

$$E_Q \left[ e^{-rt} A_T \right] = A_0.$$  

This means that assets are fairly priced, which is exactly the risk-neutral pricing approach, whatever $\sigma$. In consequence the owner is neutral towards risk. The owner does, however a strong interest in effort: ”intuitively,” at any point in time effort $\theta_t$ will shift the risk-adjusted growth rate to $\theta_t$ and so her time 0 value is $E[\exp(\int_0^T \theta_t \sigma_t dt)]$. Our focus is on the impact of effort and therefore we start by keeping the impact of changing risk-levels neutral. (Later we will see that, however, the owner is not neutral towards risk with bonus schemes, but the manager is not and the value of her claims reduce the owner’s claims.)

To determine the optimal actions of the manager and the owner we need to determine the value of their individual payouts. Many of these can later be interpreted as portfolios of suitably defined call options written on the asset value. Because the asset value is lognormal distributed, we can therefore often use the Black and Scholes (1973) option pricing formula. For an underlying security with price $S_0$ and a call option with strike (exercise) price $K$ we define

$$d_{1/2}(S_0, K, r, \sigma, T) = \ln(S_0/K) + \left(r \pm \frac{\sigma^2}{2}\right) T \frac{\sigma \sqrt{T}}{2},$$  

(5)

and use the Black and Scholes (1973) option pricing formula for the time 0 price of the call option with strike $K$

$$BS(S_0, K, r, \sigma, T) = S_0 \mathcal{N}(d_1) - Ke^{-rT} \mathcal{N}(d_2).$$  

(6)
Note that $\frac{\partial BS}{\partial K} = -e^{-rT}N(d_2)$, $\frac{\partial BS}{\partial \sigma} = S_0N'(d_1)\sqrt{T} = Ke^{-rT}N'(d_2)\sqrt{T}$. Throughout we usually apply this with $S_0 = A_0$, strike $LAe^{-\theta T}$ and maturity at time $T$.

## 3 Forms of Bonus Schemes

Throughout this paper we analyze bonus schemes as a form of incentive compensation. Each of these compensation packages is on top of a base compensation. We ignore base compensation, because it is flat and will not have any impact on risk and effort levels\(^5\). Throughout we assume that incentive compensation is offered and contracted only at time 0. In practice, there may be a new compensation package at various times before time $T$. We ignore this to compare directly the incentive effects of the different compensation forms.

Throughout we study bonus schemes with payments at a single date and assume always that this is at time $T$; we call these *single date* bonus schemes. The only exception is the so-called bonus-malus scheme that we introduce below; with this scheme there are two payments and we assume for simplicity that one is at time $T/2$ and the other at time $T$.

### 3.1 Single Date Bonus Schemes

A single date bonus scheme is a payment at time $T$ that depends on assets $A_T$. We assume there is a suitable function $f$ on the positive real line such that the size of the payment can be written $f(A_T)$. Throughout this paper we assume that for every $a > 0$ the bonus function is either differentiable with the property $0 \leq f'(a) \leq 1$ or that left- and right-hand derivatives $f'_-(a), f'_+(a)$ exist with the property that $0 \leq f'_-(a) \leq 1$ and $0 \leq f'_+(a) \leq 1$. These assumption are natural ones, because $f' \geq 0$ means that no increase in asset value will lead to a decrease in bonus payment to the manager; the situation is similar for the owners, because $f' \leq 1$ means that no increase in asset value can lead to a decrease in “payment” to the owners.

The time $T$ payouts $\Pi_M, \Pi_O$ of the manager and the owner are

$$\Pi_M = f(A_T) - \int_0^T \gamma(\theta_t)e^{r(T-t)}dt, \quad \Pi_O = A_T - f(A_T).$$

(7)

Recall that the monetary equivalent cost of the manager’s disutility will be subtracted from the value of the monetary payout to the manager. Here we carry it forward at the interest rate to

\(^5\)We ignore throughout this paper the risk the bank defaults; it would impact base compensation but this is not the risk taking that we would like to understand in this paper.
express all values at a common time, \( T \). Single-date bonus schemes differ in their functional form \( f \) which will be introduced below. Based on the time \( T \) payouts \( \Pi_M, \Pi_O \) of the manager and the owner we find the values of the manager’s and the owner’s claims

\[
V_M = e^{-rT}E_Q[\Pi_M] = e^{-rT}E_Q[f(A_T)] - \int_0^T E_Q[\gamma(\theta_t)]e^{-rt}dt, \quad (8)
\]

\[
V_0 = e^{-rT}E_Q[\Pi_O] = e^{-rT}E_Q[A_T] - e^{-rT}E_Q[f(A_T)]. \quad (9)
\]

The first form of incentive compensation is a linear claim on the asset value; we refer to this as the *stock (single-date) bonus scheme*. It pays a fixed fraction \( 0 < \beta < 100\% \), called the payout ratio, of the time \( T \) asset value.

Another form of incentive compensation is the following scheme, which we will call the *simple (single-date) bonus scheme*: the manager receives at time \( T \) the fraction \( 0 < \beta < 100\% \) of whatever exceeds a hurdle \( L \); otherwise nothing is paid or charged. As with the stock bonus scheme we call \( \beta \) the payout ratio. The payout from this bonus scheme is equal to the payment of a call option written on the bank’s assets with strike \( L \) and maturity at time \( T \):

\[
f(a) = \beta \cdot (a - L)^+. \quad (10)
\]

The payout from the simple bonus scheme resembles that of a plain vanilla call option with maturity \( T \) and strike \( L \). At the beginning of this subsection we required suitable differentiability of the payout function. The simple bonus fulfills these: it is differentiable everywhere on the positive real line except at the strike, but there it has a left-derivative of 0 and a right-derivative of 1, so that it fulfills this assumption.

The third form of incentive compensation we study is the so-called *single-date capped bonus scheme*. It pays at time \( T \) the fraction \( 0 < \beta < 100\% \) (the payout ratio) what exceeds a minimum hurdle \( L_1 \) but only up to the maximum hurdle \( L_2 \). A standing assumption throughout this paper is that \( L_2 > L_1 \). We can write this as financial payout

\[
f(a) = \beta \cdot \{(a - L_1)^+ - (a - L_2)^+\}. \quad (11)
\]

Figure 1 presents the payout profile. It is similar to the simple bonus scheme with the difference that the payout is capped to \( \beta(L_2 - L_1) \). This bonus scheme can be interpreted as a “portfolio” of two simple single-date bonus schemes: it represents a portfolio of two call options written on the bank’s assets, maturity at time \( T \): one “long” with strike \( L_1 \) and one “short” with strike \( L_2 \).
3.2 Bonus-Malus Scheme

The so-called bonus-malus model pays at two dates, here assumed at $T/2$ and $T$. The amount of the entire bonus is determined at time $T/2$; throughout we assume that this amount is the payout fraction $\beta$ of the time $T/2$ asset value which exceeds the hurdle $L$. But only a fraction $0 \leq \kappa < 100\%$ of this amount is paid out at $T/2$, i.e. time $T/2$ payment is $f_{T/2}(A_{T/2})$, where we define

$$f_{T/2}(a) = \kappa \beta (a - L)^+.$$  \hspace{1cm} (12)

The remainder (the fraction $1-\kappa$) is deferred to time $T$ and will be (partially) eliminated if the asset value $A_T$ at time $T$ falls short of $A_{T/2}$. (There is, however, no payback of that part of the bonus which has not been deferred, i.e. the deferred bonus payment cannot be negative.)

To further explore this, let us focus on the situation $A_{T/2} > L$. At time $T$, the payment is $f_T(A_T)$, where we define

$$f_T(\tilde{a}) = \left((1-\kappa)\beta(a - L)^+ - \beta(\tilde{a} - a)^-\right)^+.$$  \hspace{1cm} (13)

In the brackets, the first term $(1-\kappa)\beta(a - L)^+$ describes what has been deferred; within the brackets, we subtract from this the amount $(\tilde{a} - a)^-$ by which $\tilde{a}$ is short $a$; this amount is scaled $^6$The case $\kappa = 100\%$ means that all is paid out, so this corresponds to a simple single-date bonus scheme with payout at time $T/2$. 

$^6$The case $\kappa = 100\%$ means that all is paid out, so this corresponds to a simple single-date bonus scheme with payout at time $T/2$. 

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**Figure 1**: Time $T$ payment from capped bonus model
by the payout ratio $\beta$. (For simplicity we set the scale of the reduction equal to $\beta$.) We cannot deduct more than the deferred bonus payment and so we take the positive value of the result to describe her payout. Note that nothing is subtracted and the manager receives the entire deferred bonus if $\tilde{a} \geq a$.

**Proposition 1** We denote for $a > L$,

$$L_1(a) = a - (1 - \kappa)(a - L).$$

The bonus-malus schemes pays at time $T/2$ depending on the time $T/2$ asset value: $f(a) = \beta(a - L)^+$. At time $T$ it pays nothing if $a < L$ and for $a \geq L$ is pays $\tilde{f}(\tilde{a}) = \beta(\tilde{a} - L_1(a))^+ - \beta(\tilde{a} - a)^+.$

Figure 2 presents the time $T$ payout of the deferred part of the bonus. Proposition 1 shows that at time $T/2$ we can write the deferred part of the bonus-malus scheme as a capped bonus with payment at time $T$ with lower hurdle at $L_1(A_{T/2})$ and upper hurdle at $L_2(A_{T/2}) = A_{T/2}$. 

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Figure 2: Time $T$ payment profile of deferred bonus
4 Constant Risk and Effort Levels

4.1 The Specialized Setup

Throughout this section we look at the simplification that the manager picks the risk level \( \sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}} \) and the effort level \( 0 \leq \theta \leq \theta_{\text{max}} \) at time 0 once and keeps it at the chosen level for all times \( 0 \leq t \leq T \), i.e. the manager does not change it between 0 and \( T \), \( \theta_t = \theta, \sigma_t = \sigma \).

(The owner correctly anticipates this and desires only risk and effort levels with this property.)

This means that the risk-neutral pricing dynamics of equation (4) simplifies to:

\[
dA_t = (r + \theta \sigma) A_t dt + \sigma A_t dW_t^Q.
\]

We define the total cost of effort over the time period 0 to \( T \) by the function \( \bar{\gamma}(\theta) \) with

\[
\bar{\gamma}(\theta) = \int_0^T \gamma(\theta) e^{-rt} dt = \gamma(\theta) \frac{1 - \exp(-rT)}{r}.
\]

The properties of the cost function of equation (2) carry over, i.e.

\[
\bar{\gamma}(0) = 0, \bar{\gamma}(\theta) > 0, \bar{\gamma}'(\theta) > 0, \bar{\gamma}''(\theta) \geq 0 \quad \text{for all } 0 \leq \theta \leq \theta_{\text{max}}.
\]

With single date bonus schemes we are only interested in the distribution of asset values \( A_T \) at time \( T \). Equation (15) implies that it is lognormal distributed under \( Q \), i.e.

\[
A_T = \exp \left\{ \left( r + \theta \sigma - \frac{\sigma^2}{2} \right) T + \sigma W_T^Q \right\}.
\]

Our continuous-time setup then corresponds to a single-period discrete time setup with this distribution. Note that \( \exp(-rT)E_Q[A_T] = \exp(\theta \sigma T) \). The time 0 claim value of the manager and the owner can be written based on equations (8, 9) :

\[
V_M = e^{-rT}E_Q[f(A_T)] - \bar{\gamma}(\theta).
\]

4.2 Stock Bonus Scheme

For the stock bonus scheme, the value of the incentive contract is

\[
e^{-rT}E_Q[f(A_T)] = \beta e^{\theta \sigma T}.
\]
Theorem 2 (Stock Bonus) If the manager provides effort, then she will choose maximal risk. Assume that for all effort levels $0 \leq \theta \leq \theta_{\text{max}}$:

$$\frac{\hat{\gamma}'(\theta)}{\beta} < \sigma_{\text{max}} T e^{B_{\text{max}} T}.$$ 

Then the manager will pick the highest effort-level $\theta_{\text{max}}$ and the highest risk-level $\sigma_{\text{max}}$.

The condition in this theorem is a sufficient one such that the manager provides maximal effort but then also takes maximal risk. The condition requires the growth in disutility relative to the payout ratio $\beta$ to be less than a critical value. Note that increasing the payout ratio decreases the left-hand side and although it may not be fulfilled for some payout ratio, a higher ratio may ensure maximal effort. Finally, note that the right-hand-side in the condition is always more than $\sigma_{\text{max}} T$; so, if $\frac{\hat{\gamma}'(0)}{\beta} < \sigma_{\text{max}} T$, it is always fulfilled and the manager picks maximal effort.

Because the condition in theorem 2 is only a sufficient one, if it is violated it may well be that the manager picks another positive effort-level or may decide to provide no effort. This will depend on the functional form of the cost of disutility; there is no general expression for it. For example if $\hat{\gamma}'(0) > \sigma_{\text{max}} T e^{B_{\text{max}} \sigma_{\text{max}} T}$, then $\frac{\hat{\gamma}'(\theta)}{\beta} > \sigma_{\text{max}} T e^{B_{\text{max}} \sigma_{\text{max}} T}$ for all $\theta$ because $\hat{\gamma}'' \geq 0$; similar to the proof of the theorem, we can show that the manager does not provide effort; the risk level is then undetermined.

Theorem 3 (Stock Bonus) If $\gamma'(\theta_{\text{max}}) < \beta \sigma_{\text{max}}$, then there is a critical time $\bar{T} > 0$ such that $\theta_{\text{M}} = \theta_{\text{max}}$ for all $0 < T < \bar{T}$.

This theorem shows that if effort does not grow too much or the payout ratio is sufficiently large, then for sufficiently small time $T$, the manager picks maximal effort. While this shows that the stock bonus scheme is an appealing incentive scheme, theorem 2 states that whenever the manager picks any effort then she will also head for maximal risk. This is an undesirable property from the viewpoint of risk-management.
4.3 Simple Bonus Scheme

For the simple single date bonus scheme of equation (10), the value of the incentive contract is according to the risk-neutral pricing methodology:

$$e^{-rT}E_Q[f(A_T)] = \beta e^{-rT}E_Q[(A_T - L)^+] = \beta e^{\theta \sigma T} e^{-rT}E_Q[(A_T e^{-\theta \sigma T} - L e^{-\theta \sigma T})^+]$$ (21)

$$= \beta e^{\theta \sigma T} BS(1, L e^{-\theta \sigma T}, r, \sigma, T).$$ (22)

The optimal action and the optimal risk level of the manager will maximize the manager’s and the owner’s claim value of equations (19), respectively.

**Theorem 4 (Simple Bonus)** For the simple single date bonus scheme the manager will desire the maximal risk-level $\sigma_{max}$. Assume that for all effort levels $0 \leq \theta \leq \theta_{max}$:

$$\frac{\gamma'(\theta)}{\beta} < \sigma_{max} T e^{\theta \sigma_{max} T} N\left(d_1(1, L e^{-\theta \sigma_{max} T}, r, \sigma_{max}, T)\right).$$

Then the manager will pick the highest effort-level $\theta_{max}$.

The right-hand side in this condition is positive, so it requires that the monetary equivalent cost of disutility in relation to the payout ratio $\beta$ does not grow too fast as effort increases. If we evaluate the right-hand side at $\theta = 0$, the left-hand side at $\theta_{max}$, and if the resulting left-hand side is less than the resulting right-hand side, then the condition holds for all $0 \leq \theta \leq \theta_{max}$ because the right-hand side and the left-hand side are increasing in $\theta$.

Because the payout ratio is in the denominator, increasing the payout ratio may ultimately incentivize the manager to provide maximal effort even if a lower payout ratio does not. Put differently, to ensure maximum effort, the payout ratio can be chosen as the smallest $\beta$ which still ensures the condition.

Note that the condition in the theorem is only a sufficient one, i.e. when it is violated the theorem does not specify how the manager decides. In general, this depends on the minimum and maximum choices for risk, the strike and the time $T$; we refrain from a detailed analysis as it is both tedious and does not provide particular insight. It is, however, illustrative for our later continuous-time analysis to study what happens when time $T$ is small:
Theorem 5 (Simple Bonus) Assume that \( \gamma'(\theta_{\text{max}}) < \beta \sigma_{\text{max}}/2 \). Then there is a critical time \( \bar{T} > 0 \) such that for all \( 0 < T < \bar{T} \)

\[
\theta^*_M = \begin{cases} 
\theta_{\text{max}} & ; \text{if } L \geq 1 \\
0 & ; \text{otherwise}
\end{cases}
\]

This theorem shows that the simple bonus scheme may not fulfill its purpose. For short maturities, if \( L \geq 1 \), the option will likely pay off and (under the stated assumption on the cost of effort) pay enough to incentivize the manager for full effort; this result should not come as a surprise given our results. However, surprising is that if \( L \) is smaller than one, the manager will consider it unlikely that the bonus pays out and not provide any effort. This will play a crucial role in our continuous-time section 5.

4.4 Capped Bonus Scheme

The capped bonus payment can be interpreted as a portfolio of two simple bonus schemes, see our discussion in subsection 3.1. Based on the simple bonus valuation formula of equation (22) we can derive the value of the capped bonus contract:

\[
e^{-rT}E_Q[f(A_T)] = \beta e^{\theta T} \left\{ BS(1, L_1 e^{-\theta T}, r, \sigma, T) - BS(1, L_2 e^{-\theta T}, r, \sigma, T) \right\}.
\]

The optimal action and the optimal risk level of the manager will maximize the manager’s claim value of equation (19). For further analysis define for \( 0 \leq \theta \leq \theta_{\text{max}} \) the function \( \sigma_0 \) by setting

\[
\sigma_0(\theta) = \theta + \sqrt{\theta^2 + \frac{\ln(L_1 L_2)}{T} - 2r}.
\]

There are two (potentially counteracting) forces on the bonus claim value through an increase in the risk-level. When \( \theta > 0 \), there is a positive force through the asset drift because it increases the expected asset value as the drift beyond the riskfree rate enters as \( \theta \sigma T \). However, even ignoring the drift \( \theta \sigma T \), there is another force that is entirely due to the dispersion of asset values: intuitively, when the current asset value is above the upper hurdle, the intrinsic bonus claim value is at its maximum, and so an increase in the risk merely increases the chance of ending up below that upper hurdle \( L_2 \); the bonus claim value therefore decreases as risk increases. When the current asset value is below the lower hurdle, an increase in risk initially increases the chance of larger payouts and so increases the claim value; however as we increase risk, the upper hurdle limits payouts and so beyond a critical value the claim value may decrease.
These two forces determine if the maximal risk-level or a smaller one is optimal for the manager. The contribution to the capped bonus claim value from the second force has a maximum at $\sigma_0(\theta)$ for given effort level $\theta$. Because the first force always makes a positive contribution when $\theta > 0$, the optimal risk-level for the manager must be larger. There is no general closed-form expression for $\sigma^*_M$. The appendix proves:

**Theorem 6 (Capped Bonus)** Assume $L_1L_2 > e^{2rT}$. Then $\sigma_0(\theta)$ is a strictly positive, strictly increasing function on the interval $0 \leq \theta \leq \theta_{\text{max}}$. The manager’s risk-level $\sigma^*_M$ has the property that $\sigma^*_M \geq \sigma_0(\theta^*_M)$ if $\sigma_0(\theta^*_M) < \sigma_{\text{max}}$ and $\sigma^*_M = \sigma_{\text{max}}$ if $\sigma_0(\theta^*_M) \geq \sigma_{\text{max}}$. Furthermore, if the manager decides on zero effort, then the optimal risk-level is $\sigma_0(0)$.

Assume that for all $\sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}}$ and all $0 \leq \theta \leq \theta_{\text{max}}$:

$$\frac{\gamma'(\theta)}{\beta} < \sigma e^{\theta \sigma T} \left( N \left( d_2(1, L_1 e^{\theta \sigma T}, r, \sigma, T) \right) - N \left( d_2(1, L_2 e^{\theta \sigma T}, r, \sigma, T) \right) \right).$$

Then the manager will pick the maximal effort-level $\theta^*_M = \theta_{\text{max}}$.

Under assumption $L_2 > L_1$ we must have $L_1 > 1$ to ensure $L_1L_2 > e^{2rT}$. Typically, $L_1 > 1$ and the upper hurdle $L_2$ will typically be chosen sufficiently larger than 1. Therefore, the assumption $L_1L_2 > e^{2rT}$ is not restrictive. This theorem tells us that the risk-level will be larger than $\sigma_0$, depending on the effort-level $\theta^*_M$. Because $\sigma_0$ is an increasing function of effort, it means it is always more than $\sigma_0(0) = \sqrt{\ln(L_1L_2)/T - 2r}$.

The lower boundary on the optimal risk level, $\sigma_0$, has interesting interpretations: as $L_2$ increases the risk-level $\sigma_0$ increases; at one point the calculated value for $\sigma_0$ would grow beyond $\sigma_{\text{max}}$ and the manager would pick the maximal risk-level. This matches our intuition: as $L_2$ tends to infinity, the capped bonus becomes more and more like a simple bonus.

Note that condition (25) is positive on the right-hand side, because $L_2 > L_1$. Differently to the corresponding simple bonus theorem 4, this one here needs to hold for all risk- and effort levels. As with the simple bonus, we can potentially increase the payout ratio to ensure that the condition here holds and the manager provides maximal effort. As with the simple bonus theorem 4, we want to stress that the condition here is only a sufficient one and does not permit us to draw conclusions on effort, when it is violated. A general analysis is beyond our scope in this paper.
It is interesting that the theorem states that without effort the manager picks the risk-level \( \sigma_0 \), i.e. the manager keeps risk under control. From a theoretical perspective this explains the attraction of capped bonus schemes in practice.

Applying the theorem, as we decrease time \( T \) and let it tend to 0, \( \ln(L_1 L_2)/T \) will increase, so that ultimately \( \sigma_0(0) \) exceeds \( \sigma_{\text{max}} \), and therefore \( \sigma_0(\theta) > \sigma_{\text{max}} \) for all \( \theta \). This means that for sufficiently small \( T \), \( \sigma^*_M = \sigma_{\text{max}} \), whatever the hurdle rates, the interest rate and whatever the effort level. Similarly, as \( T \) tends to zero, the conditions on \( \gamma \) will be violated, because the right-hand side in the condition (equation (25)) tends to 0 as \( T \) tends to 0. As this is only a sufficient condition, the theorem will not tell us any more what effort and risk levels the manager picks. The appendix proves:

**Theorem 7 (Capped Bonus)** Assume that \( \gamma'(\theta_{\text{max}}) < \beta/2 \). Then there is a critical time \( \bar{T} > 0 \) such that for all \( 0 < T < \bar{T} \)

\[
\theta^*_M = \begin{cases} 
\theta_{\text{max}} & \text{if } L_1 \leq 1 \leq L_2 \\
0 & \text{otherwise} 
\end{cases}
\quad \text{and} \quad \sigma^*_M = \begin{cases} 
\sigma_{\text{max}} & \text{if } L_2 > 1 \\
\sigma_{\text{min}} & \text{if } L_2 < 1 
\end{cases}
\]

The result is surprising for several reasons. First of all, it is not in the interest of incentive compensation that the manager only provides effort when the asset value is within a range, i.e. larger than the lower hurdle \( L_1 \) and smaller than the upper hurdle \( L_2 \). Furthermore, the manager will reduce risk to a minimum if the asset value is larger than the upper hurdle, but below that same hurdle, the risk not controlled at all as the manager picks maximal risk. This points to a problem we will encounter in continuous-time.

### 4.5 Bonus-Malus Scheme

Using the definition in equation (16) the time 0 claim value of the manager then can be written based on equations (8, 9):

\[
V_M = -\bar\gamma(\theta) + e^{-rT/2}E_Q[f(A_{T/2})] + e^{-rT}E_Q[f(A_T)],
\]

(26)

In subsection 3.2 we characterized the bonus-malus scheme as a capped bonus where lower and upper hurdle depend on the time \( T/2 \) asset value. If we denote the asset value by \( a \) and assume it is larger than \( L \), then the lower hurdle is \( L_1(a) = a - (1 - \kappa)(a - L) \) and the upper hurdle is \( L_2 = a \). (If \( a < L \) the bonus pays nothing.) Based on the results for the capped
bonus scheme in the previous subsection we then find that the time $T/2$ bonus-malus value is conditional on $A_{T/2} > L$:

$$e^{-rT/2}E_Q\left[f(A_T)\bigg| A_{T/2} > L\right] = \beta e^{\theta \sigma T} BS\left(A_{T/2}, (A_{T/2} - (1 - \kappa)(A_{T/2} - L)) e^{-\theta \sigma T}, r, \sigma, T/2\right) - \beta e^{\theta \sigma T} BS\left(A_{T/2}, A_{T/2} e^{-\theta \sigma T}, r, \sigma, T/2\right).$$

Using this, the value of the incentive contract is then:

$$\beta e^{-rT/2}E_Q\left[\kappa \left(A_{T/2} - LA_{T/2}\right)^+ + e^{-rT/2}E_Q\left[f(A_T)\bigg| A_{T/2} > L\right]\right]$$

Unfortunately, we could not find a closed-form expression for this. Therefore we need to resort to numerical integration and specific parameters. This will be studied further in subsection 6.1.

5 Continuously Adjusted Risk and Effort Levels

Throughout this section the manager and the owner continuously adjust their optimal effort and risk-levels. In the following we view them as functions $\theta^*_M(t, a), \sigma^*_M(t, a)$ for the manager and functions $\theta^*_O(t, a), \sigma^*_O(t, a)$ for the owner that depend on time $t$ and the current asset value $a$.

5.1 Manager’s Incentives under General Bonus Schemes

Throughout this subsection we study the general implications of bonus schemes on risk and effort levels. For this we assume the bonus function is twice continuously differentiable on the positive real line. The current bonus schemes presented in section 3 are not twice continuously differentiable at all positive real numbers but we can use the treat them analogously to the analysis in this subsection.

Often we need the inverse of $\gamma'$ and define it as follows. By assumption $\gamma'' \geq 0$ and so $\gamma'(0) \leq \gamma'(\theta_{max})$. If $\gamma'(0) < \gamma'(\theta_{max})$ there is an inverse function, denoted $(\gamma')^{-1}$ of the function $\gamma'$; otherwise $\gamma'$ is constant and if we look for the inverse at this constant, then the result is undetermined: it is the entire interval $[0, \theta_{max}]$.

**Theorem 8** Assume the bonus function is twice continuously differentiable on the positive real line. Define for any given asset value $a$ at a time $t$

$$F(t, a) = (r + \theta \sigma)a f'(a) + \frac{1}{2} \sigma^2 a^2 f''(a) - \gamma(\theta)e^{-\tau t}.$$
For any $0 \leq t \leq T, 0 < a$, the manager will choose the risk- and effort levels $\sigma^*(t,a), \theta^*(t,a)$ such that $F(t,a)$ is maximized.

This is our basic workhorse to determine risk and effort levels. The first- and second order derivatives w.r.t. $\theta$ and $\sigma$ are

$$\frac{\partial F}{\partial \theta} = \sigma f'(a) - \gamma'(\theta)e^{-rt}, \quad \frac{\partial^2 F}{\partial \theta^2} = -\gamma''(\theta)e^{-rt}$$

$$\frac{\partial F}{\partial \sigma} = \theta f'(a) + \sigma a^2 f''(a), \quad \frac{\partial^2 F}{\partial \sigma^2} = a^2 f''(a). \quad (27)$$

The following three theorems characterize the optimal risk and effort levels for the manager for given time $t$ and asset value $a$ at that time. They look at the local behavior of the bonus function $f$ at the asset value and study separately the locally convex bonus function (theorem 9), the locally linear bonus function (theorem 10) and the locally concave bonus function (theorem 11).

**Theorem 9 (Locally Convex Bonus Function)** Assume a single date bonus scheme with a bonus function that is twice continuously differentiable $f \in C^2$ and for which $f''(a) > 0$ at the current asset value $a$. Then $\sigma^*_M(t,a) = \sigma_{\text{max}}$ and

$$\theta^*_M(t,a) = \begin{cases} 0 & \text{if } \sigma_{\text{max}}af'(a)e^{rt} > \gamma'(0) \\ \theta_{\text{max}} & \text{if } \sigma_{\text{max}}af'(a)e^{rt} < \gamma'(\theta_{\text{max}}) \\ (\gamma')^{-1}(\sigma_{\text{max}}af'(a)e^{rt}) & \text{if } \gamma'(0) \leq \sigma_{\text{max}}af'(a)e^{rt} \leq \gamma'(\theta_{\text{max}}). \end{cases}$$

This characterizes the optimal effort level as follows: The term $\sigma_{\text{max}}af'(a)e^{rt}$ can be smaller than $\gamma'(0)$, larger than $\gamma'(\theta_{\text{max}})$ or lie in between. In the latter case and if $\gamma'(0) < \gamma'(\theta_{\text{max}})$ the inverse $(\gamma')^{-1}$ of $\gamma'$ then gives the optimal $\theta^*$ as the unique $\theta^*$ for which $\sigma_{\text{max}}af'(a)e^{rt} = \gamma'(\theta^*)$; if, however, $\gamma'(0) = \sigma_{\text{max}}af'(a)e^{rt} = \gamma'(\theta_{\text{max}})$, then the effort level is undetermined.

**Theorem 10 (Locally Linear Bonus Function)** Assume a single date bonus scheme with a bonus function that is twice continuously differentiable $f \in C^2$ and for which $f''(a) = 0$ at the current asset value $a$. If $\sigma_{\text{max}}af'(a)e^{rt} \leq \gamma'(0)$, then the manager will not provide effort ($\theta^*_M = 0$) and the risk-level is undetermined. However, if $\sigma_{\text{max}}af'(a)e^{rt} > \gamma'(0)$ then the manager picks the maximum risk level ($\sigma^*_M(t,a) = \sigma_{\text{max}}$) and

$$\theta^*_M(t,a) = \begin{cases} \theta_{\text{max}} & \text{if } \sigma_{\text{max}}af'(a)e^{rt} > \gamma'(\theta_{\text{max}}) \\ (\gamma')^{-1}(\sigma_{\text{max}}af'(a)e^{rt}) & \text{if } \gamma'(0) \leq \sigma_{\text{max}}af'(a)e^{rt} \leq \gamma'(\theta_{\text{max}}) \end{cases}.$$
In the theorem, when the manager does not provide effort, the risk-level is undetermined. Intuitively, with no effort, local linearity corresponds to local risk-neutrality; we have seen this earlier in our analysis of the stock bonus scheme with constant (over time) risk and effort levels, see subsection 4.2.

**Theorem 11 (Locally Concave Bonus Function)** Assume a single date bonus scheme with a bonus function that is twice continuously differentiable \( f \in C^2 \) and for which \( f''(a) < 0 \) at the current asset value \( a \). We define the set

\[
S_t = \left\{ \theta \left| -\frac{(f'(a))^2}{f''(a)} - \gamma'(\theta)e^{-rt} = 0 \text{ and } -\frac{f'(a)}{af''(a)} \leq \sigma_{max} \right. \right\}.
\]

If this set \( S_t \) is empty, then the manager will pick \( \theta^*_M(t,a) = 0 \) and \( \sigma^*_M(t,a) = \sigma_{min} \). If the set \( S_t \) is not empty, then we set \( \sigma(\theta) = \min \left\{ \sigma_{max}, -\frac{f'(a)}{af''(a)} \right\} \) for all \( \theta \in S \); the optimal \( \theta^*_M(t,a), \sigma^*_M(t,a) \) is that \( \theta, \sigma(\theta) \) which maximizes

\[
(r + \theta \sigma)a f'(a) + \frac{1}{2} \sigma^2 a^2 f''(a) - \gamma(\theta)e^{-rt}.
\]

Unfortunately, there is no general expression for the optimal effort and risk levels. The origin of this is that the manager may pick an interior solution: a risk and/or an effort level between their minimal and maximal values. Potentially there can be many interior solutions to the first-order conditions in equations (27, 28); we therefore need to determine which of these maximizes the manager’s claim value.

This theorem will come handy in subsection 5.4 when we describe bonus functions for which the manager picks target risk and effort levels. (These functions are locally concave.) Furthermore, in section 6 where we illustrate our results it will not be hard to apply this theorem.

It has been argued that the kinks in bonus functions induce managers to suboptimal effort, see, e.g. Jensen, Murphy, and Wruck (2004). We see here that not the kinks are the issue, but that the different slopes are important: we need to distinguish locally convex, locally linear and locally concave behavior. Each induces the manager to different tradeoffs with regard risk and effort. In subsection 5.4 we will pursue this analysis to design bonus schemes such that the manager picks target risk and effort levels.
5.2 Manager’s Incentives with Current Bonus Schemes

Applying theorem 10 we directly find:

**Theorem 12 (Stock Bonus)** For the stock bonus scheme, if $\sigma_{max} a e^{rt} \leq \gamma'(0)$, the manager will not provide effort ($\theta^*_M = 0$) and the risk-level is undetermined. However, if $\sigma_{max} a e^{rt} > \gamma'(0)$ then the manager picks the maximum effort and risk level $\sigma^*_M(t,a) = \sigma_{max}$ and and

$$\theta^*_M(t,a) = \begin{cases} \theta_{max} & \text{if } \sigma_{max} a e^{rt} > \gamma'(\theta_{max}) \\ (\gamma')^{-1}(\sigma_{max} a e^{rt}) & \text{if } \gamma'(0) \leq \sigma_{max} a e^{rt} \leq \gamma'(\theta_{max}) \end{cases}$$

While this theorem states that risk is undetermined if the manager does not provide effort, this should not come as a surprise. When the manager does not provide effort, there is no additional contribution to the drift beyond fair pricing: the value is always equal to the payout ratio of the current asset value. Our previous analysis shows that only when the bonus function is partially non-linear, effort can be zero and the manager has clear preferences on the risk-level; similarly we will see this below for the simple and the capped bonus.

**Theorem 13 (Simple Bonus)** For the simple bonus scheme the manager will provide no effort $\theta^*_M(t,a) = 0$ when $a < L$ and pick maximal risk $\sigma^*_M(t,a) = \sigma_{max}$. For $a \geq L$, the manager will choose the maximal risk-level $\sigma^*_M(t,a) = \sigma_{max}$ and

$$\theta^*_M(t,a) = \begin{cases} 0 & \text{if } \sigma_{max} a e^{rt} < \gamma'(0) \\ \theta_{max} (\gamma')^{-1}(\sigma_{max} a e^{rt}) & \text{if } \sigma_{max} a e^{rt} > \gamma'(\theta_{max}) \\ \theta_{max} & \text{if } \gamma'(0) \geq \sigma_{max} a e^{rt} \geq \gamma'(\theta_{max}) \end{cases}$$

Note that $\gamma'(\theta_{max}) \geq \gamma'(0) > 0$ by assumption. Therefore, for some $t$, the condition about $\sigma_{max} a - \gamma'(0)e^{-rt}$ means that there is a lower cutoff $\bar{a}_l(t) = \gamma'(0)e^{-rt}/\sigma_{max}$ such that this is negative for all $a < \bar{a}_l(t)$; similarly there is an upper cutoff $\bar{a}_u(t) = \gamma'(\theta_{max})e^{-rt}/\sigma_{max}$ such that $\sigma_{max} a - \gamma'(\theta_{max})e^{-rt}$ is positive for all $a > \bar{a}_u(t)$. This means that if $\bar{a}_l(t) > L$, then the manager will pick no effort $\theta^*_M(t,a) = 0$ for all $L \leq a < \bar{a}_l(t)$. Similarly, if $\bar{a}_u(t) > L$ then the manager will pick maximal effort $\theta^*_M(t,a) = \theta_{max}$ for all $a > \bar{a}_u(t)$.

This result is interesting, because there may be asset values larger than the hurdle for which the manager provides no effort. However, for all values larger than $\max\{L, \bar{a}_u(t)\}$ the manager will provide maximal effort. However, this result is driven by a payout ratio that is too small: increasing the payout ratio we can decrease the upper cutoff and once it is less than $L$ the
manager provides maximal effort for all $a > L$. Note that this result is in line with our limit result in theorem 5 because the condition there implies $a_u(t) < 1/2$ and in this case we find also here that the manager provides maximal effort for $a \geq L$.

**Theorem 14** For the capped bonus scheme the manager will take maximal risk $\sigma^*_M(t, a) = \sigma_{\text{max}}$ for $a < L_2$ and minimal risk $\sigma^*_M(t, a) = \sigma_{\text{min}}$ for $a > L_2$. She provides no effort $\theta^*_M(t, a) = 0$ when $a < L_1$ or $a > L_2$; for $L_1 < a < L_2$ the manager will choose

$$
\theta^*_M(t, a) = \begin{cases} 
0 & \text{if } \sigma_{\text{max}}ae^{rt} < \gamma'(0) \\
\theta_{\text{max}} & \text{if } \sigma_{\text{max}}ae^{rt} > \gamma'(\theta_{\text{max}}) \\
(\gamma')^{-1}(\sigma_{\text{max}}ae^{rt}) & \text{if } \gamma'(0) \leq \sigma_{\text{max}}ae^{rt} \leq \gamma'(\theta_{\text{max}})
\end{cases}.
$$

The results here are analogous to that of the simple bonus scheme (theorem 13). For the same lower and upper cutoffs, we then find for values $L_1 < a < L_2$ that are below (above) these cutoffs the same results about no effort (maximal effort). Also, as noted there, this means that the payout ratio is too small. Furthermore, note that this result is in line with the limit result of theorem 7.

An interesting implication of the theorem is that when $\sigma_{\text{max}} = 0$, i.e. if the manager can choose zero risk, then the manager will pick zero risk for $a < L_1$ and $a > L_2$. In consequence, the stock cannot go beyond $L_2$ if we start at $1 < L_2$. (If we start at $1 > L_2$, i.e. above the upper hurdle, then we stay at this level.) This is against the interest of incentive compensation: the assets cannot start at below the hurdle $L_2$ and then grow beyond it; the capped bonus scheme limits the growth of the stock.

Proposition 1 characterized the bonus-malus scheme at time $T/2$ as a capped bonus scheme. We denote the time $T/2$ asset value by $A_{T/2} = a$. Assume $a > L$ and recall that the bonus-malus scheme takes lower hurdle $L_1 = a - (1 - \kappa)(a - L)$ and upper hurdle $L_2 = a$. Theorem 14 then characterizes the manager’s effort and risk-levels over the time interval $T/2$ to $T$; note that we are exactly at the upper hurdle at time $T/2$ and therefore if the asset value increases lightly the manager will reduce risk to the minimum and provide no effort at all; this is not a very desirable situation for an incentive scheme.

**Theorem 15** With the bonus-malus scheme, if $a > L$ denotes the conditional time $T/2$ asset
value, then we define

\[ g_1(a) = \beta \int_{T/2}^{T} e^{r(t-T/2)} \left\{ \mathcal{N} \left( d_1 \left( a, L_1, r, \sigma_{\text{max}}, t - \frac{T}{2} \right) \right) - \mathcal{N} \left( d_1 \left( a, L_2, r, \sigma_{\text{max}}, t - \frac{T}{2} \right) \right) \right\} dt, \]

\[ g_2(a) = \gamma(\theta_{\text{max}}) \int_{T/2}^{T} \left\{ \mathcal{N} \left( d_2 \left( a, L_1, r, \sigma_{\text{max}}, t - \frac{T}{2} \right) \right) - \mathcal{N} \left( d_2 \left( a, L_2, r, \sigma_{\text{max}}, t - \frac{T}{2} \right) \right) \right\} dt. \]

The function \( g_1 \) describes the time \( T/2 \) value of the manager’s claim that exceeds her intrinsic value, given her optimal effort between \( T/2 \) and \( T \); the function \( g_2 \) describes the total monetary equivalent value from corresponding utility loss. We can therefore write the time \( T/2 \) value of the manager’s claim for all \( a > 0 \) as

\[ \tilde{f}(a) = 1_{a>L} \{ \beta(a - L) + g_1(a) - g_2(a) \}. \]

This result has the interesting interpretation that the time \( T/2 \) value of the manager’s claim in the bonus-malus scheme is not only the simple bonus; we add the additional value \( g_1 \) from deferring the bonus over its intrinsic claim value but we need to subtract for the manager the cost \( g_2 \) of providing effort. Note that \( g_1(L) - g_2(L) = 0 \).

We can then use Theorems 9–11 using \( \tilde{f} \) instead of \( f \) to determine effort and risk-levels of the manager. For this it is imperative to determine the curvature of \( g_1 - g_2 \). We know that both functions are concave, but the difference can therefore take different forms depending on the size of \( \gamma(\theta_{\text{max}}) \) and can take at least the following three forms for \( a > L \) as we increase \( a \): first, decreasing convex; second, increasing concave; third, hump-shaped, i.e. initially increasing concave, then reaching a maximum and then decreasing concave. We postpone further analysis to the section 6, where we specify parameters to illustrate our results.

5.3 Desired Actions by Owner

Theorem 16 (Desired Effort and Risk by The Owner) For a single date bonus scheme with a bonus function that is twice continuously differentiable \( f \in C^2 \), the owner will always desire the maximal effort-level \( \theta^*_O(a) = \theta_{\text{max}} \) and the risk-level

\[ \sigma^*_O(a) = \begin{cases} \min \left\{ \sigma_{\text{max}}, -\frac{\theta_{\text{max}}}{a f''(a)} \right\} & \text{if } f''(a) < 0, \\ \sigma_{\text{max}} & \text{if } f''(a) \geq 0. \end{cases} \]
Note that the owner always desires the maximal effort level. The desired risk-level depends on the curvature of the bonus scheme at the current asset value: if it is positive, the bonus scheme is locally convex and the owner desires the maximal risk-level; if it is negative, the bonus scheme is locally concave and the owner needs to trade off any increase coming from the drift \((1 - f')\) against the “aversion” towards risk due to concavity \((f'')\). Note that \(-\theta_{\text{max}} \frac{1-f'(a)}{af''(a)}\) is always positive, when \(f''(a) < 0\), but it may be larger than the maximal risk-level and then we pick \(\sigma_{\text{max}}\).

When the bonus function is locally concave \(f''(a) < 0\), then in the absence of constraints on maximal risk-levels and under the assumption the manager picks maximal effort, then theorems 11 and 16 tells us:

\[
\sigma^*_O(a) = -\theta_{\text{max}} \frac{1-f'(a)}{af''(a)} = \theta_{\text{max}} \frac{f'(a)}{af''(a)} - \frac{\theta_{\text{max}}}{af''(a)} = -\theta_{\text{max}} - \sigma^*_M(a).
\]

Note that the first term is positive. Although this relies on strong assumptions, it points to a fundamental disagreement between manager and owner. We will not pursue this, however, not further.

**Theorem 17** For the stock, the simple and the capped bonus schemes the owner will desire maximal risk and effort levels \(\theta^*_O(a) = \theta_{\text{max}}, \sigma^*_O(a) = \sigma_{\text{max}}\) for all \(a > 0\).

### 5.4 Developing Appropriate Bonus Schemes

In this subsection we develop bonus schemes that keep risk under control and provide incentives for effort. While there are many ways to keep risk under control we will develop in this subsection only one where the risk-level is constant whatever the asset value and whatever the time. This one is not only of particular interest, but also illustrates the approach to develop bonus schemes with other properties the reader may find suitable.

**Theorem 18** Assume the manager has decided on some effort level \(\bar{\theta}\) and \(\bar{\sigma} > 0\) is given. If \(\frac{\bar{\theta}}{\bar{\sigma}} < 1\), the manager chooses for all asset values a constant risk-level equal \(\bar{\sigma}\), if and only if the bonus function is of power form \(f(a) = \beta a^{1-\bar{\theta}/\bar{\sigma}}\) for some \(\beta > 0\). If \(\frac{\bar{\theta}}{\bar{\sigma}} > 1\), the manager chooses for all asset values a constant risk-level equal \(\bar{\sigma}\), if and only if the bonus function is of power form \(f(a) = -\beta a^{1-\bar{\theta}/\bar{\sigma}}\) for some \(\beta > 0\).
While the theorem provides a clear answer regarding the form of the bonus function, it raises two issues that need to be addressed. The first is that the bonus function in this theorem depends on the chosen effort level and we still need to check that the manager chooses the target effort level $\theta^*_M = \bar{\sigma}$. The second is that if we would like to see $\frac{\bar{\theta}}{\bar{\sigma}} > 1$, then the bonus function is negative and therefore cannot define a bonus scheme. We address these issues one at a time.

Initially, we address the first issue and for this we define for given risk level $\bar{\sigma}$ and effort level $\bar{\theta}$ the critical asset value function over time

$$\bar{a}(t) = \frac{\gamma'(\bar{\theta})e^{-rt}}{\beta(\bar{\sigma} - \bar{\theta})}.$$

**Theorem 19** Assume current time is $t$ and the manager has decided on risk level $\bar{\sigma} > 0$. Furthermore a target effort level $\bar{\theta}$ is given. The bonus function $f$ is as in theorem 18. If $\gamma'(0) < \gamma'(\theta_{\text{max}})$, the manager chooses effort level $\bar{\theta}$ for the asset value $\bar{a}(t)$; effort $\theta^*_M \leq \bar{\theta}$ for smaller asset values $a < \bar{a}(t)$ and $\theta^*_M = 0$ for sufficiently small asset values; effort $\theta^*_M \geq \bar{\theta}$ for larger asset values $a > \bar{a}(t)$ and $\theta^*_M = \theta_{\text{max}}$ for sufficiently large asset values. If $\gamma'(0) = \gamma'(\theta_{\text{max}})$, the manager chooses no effort for $a < \bar{a}(t)$, and maximal effort level $\theta_{\text{max}}$ for all larger asset values $a \geq \bar{a}(t)$.

Theorem 19 implies that it is not reasonable to set a target effort level smaller than $\theta_{\text{max}}$, because we can only achieve it at one asset value at most. However, the owner desires always maximal effort, see theorem 16; therefore we assume that the target effort level is $\bar{\theta} = \theta_{\text{max}}$. However, even then we cannot achieve this for all asset value, it is not feasible for $a < \bar{a}(t)$.

Theorems 18 and 19 together imply that there is no single date bonus scheme for which we can set constant target risk and effort levels that the manager will choose for all asset values.

When the manager does not provide effort for $a < \bar{a}(t)$, then it does not make sense to compensate her for those values and we will set the bonus payout to zero. A problem is then that the bonus function “jumps” at $\bar{a}(t)$, i.e. the payout is zero for values smaller but it is more than $\bar{a}^{1-\bar{\theta}/\bar{\sigma}}$ for values larger than $\bar{a}(t)$; to resolve this we subtract that value. Another problem is that the bonus function so defined would depend on time $t$; to resolve this note that, because $\bar{\sigma} - \bar{\theta} < 0$, we have $\bar{a}(t) \leq \bar{a}(T)$ for all $0 \leq t \leq T$. Overall, we therefore define for $\frac{\bar{\theta}}{\bar{\sigma}} < 1$ the bonus function

$$f(a) = \begin{cases} 
0 & \text{if } a \leq \bar{a}(T) \\
\beta a^{1-\bar{\theta}/\bar{\sigma}} - \beta \bar{a}^{1-\bar{\theta}/\bar{\sigma}} & \text{if } a \geq \bar{a}(T) 
\end{cases}.$$
Figure 3: Shape of the bonus functions that ensure the manager will choose a target risk level.

It remains to address the second issue that for $\frac{\theta}{\sigma} > 1$, the bonus function is negative and therefore cannot define a bonus scheme. To resolve this, recall that according to theorem [19] we cannot have the manager choose the maximal effort level for $a < \bar{a}(t)$, but then we set the payout from the bonus function equal to zero for all those values and let payouts start at 0 from thereon. Therefore, we define the bonus function

$$f(a) = \begin{cases} 0 & \text{if } a \leq \bar{a}(T) \\ \beta \bar{a}^{1-\theta/\sigma} - \beta a^{1-\theta/\sigma} & \text{if } a \geq \bar{a}(T) \end{cases}.$$ 

Note that $f(\bar{a}) = 0$ and $f \geq 0$ for all $a$ because we added the value at $\bar{a}$ of the bonus function in theorem [18] to that function. This addition does not affect the derivatives and therefore does not affect the risk and effort levels the manager chooses. Therefore we have resolved the negativity problem of the bonus function while keeping the incentive properties.

Figure 3 presents the shape of bonus functions that has the manager choose the target risk level. Note that the shape is like a smoothed capped bonus scheme. We look closer at the functional forms in the parameterization section [6].
6 Parameterization

We assume here the linear cost function
\[ \gamma(\theta) = \theta c \text{ for all } 0 \leq \theta \leq \theta_{\text{max}} \text{ so that } \gamma'(\theta) = c. \] (29)

Note that \( \gamma'(0) = \gamma'(\theta_{\text{max}}) \) because \( \gamma' \) is constant. For illustration we use the following parameters: \( \theta_{\text{max}} = 1, \sigma_{\text{min}} = 0, \sigma_{\text{max}} = 0.5, r = 5\%, T = 2 \) and \( L = 1, c = 0.01, \beta = 5\% \). When we study the bonus-malus we assume \( \kappa = 1/2 \), i.e. half of the time \( T/2 \) bonus is deferred. For the capped bonus we assume \( L_2 = 2 \).

6.1 Constant Risk and Effort Levels

We calculate
\[ \frac{\gamma'(\theta)}{\beta} = \frac{c}{\beta} \left( 1 - \exp(-rT) \right). \]

For our parameter choices the condition in theorem 2 for the stock bonus and the condition in theorem 4 for the simple bonus are fulfilled: for both the left-hand side is the same, independent of \( \theta \) and equal to 0.3807; the minimal value for the right-hand side in the conditions is 2.7183 for the stock and 0.5562 for the simple bonus. We conclude therefore that for these bonus schemes the manager choose maximal effort and maximal risk. The condition for the capped bonus in theorem 6 is not fulfilled, because the minimal value on the right-hand side is 0.0020; however, this is only a sufficient condition. Therefore we carry out a numerical analysis to determine optimal effort and risk levels. We proceed similarly for the bonus-malus because we do not have closed-form expressions.

Table 1 presents the the manager’s optimal risk and effort levels, the bonus value (derived as the discounted expected payout) as well as the manager’s claim value, i.e. bonus value less the monetary cost of disutility. We look in Panel A at the stock and the simple bonus scheme with different hurdles. Panels B and C look at capped bonus schemes with lower hurdle \( L_1 = 1 \) and lower hurdle \( L_1 = 1.25 \), respectively. Panels D and E look at the bonus-malus scheme bonus with \( \kappa = 0 \) (entire bonus is deferred at \( T/2 \) to \( T \)) and with \( \kappa = 1/2 \) (half the time \( T/2 \) bonus is deferred to \( T \)), respectively.

Panel A shows for the stock and the simple bonus schemes that the manager provides maximal effort and takes maximal risk, as expected. The claim values are largest for the stock and decrease...
### Panel A: Basic Schemes

<table>
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<tr>
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<td>$\theta_{\text{max}}$</td>
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### Panel B: Capped Bonus $L_1 = 1$

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<th>$L_2 = 1.75$</th>
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### Panel C: Capped Bonus $L_1 = 1.25$

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### Panel D: Bonus-Malus $\kappa = 0$

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### Panel E: Bonus-Malus $\kappa = 0.5$

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<tr>
<td>$\sigma_M^*$</td>
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<td>0.0012</td>
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</table>

Table 1: Optimal effort and risk levels as well as bonus claim value without cost of disutility and with ($V_M$); different bonus schemes and parameters; constant risk and effort level over time.
as the hurdle $L$ increases. For the capped bonus in Panel A we see that for many upper hurdles
the manager will not provide effort, but that, fortunately risk is not maximal. We know that
without effort the optimal risk-level is $\sigma_0(0)$ of equation (24), see theorem (6). For reference we
report that with $L_1 = 1, L_2 = 2$ the value $\sigma_0(0) = 0.4966$ and $\sigma_0(\theta_{max}) = 2.1165$. This shows
that it may be deceiving to expect risk to be under control with capped bonus schemes. In
Panel D we find similar results. Panels D and E for the bonus-malus scheme do not provide
much consolation: here we also find always maximal risk-levels and either maximal or no effort.
Particularly interesting is that increasing the hurdle $L$, at $L = 2$ the manager does not provide
effort, while she does for smaller hurdle. The intuition for this result is that with higher hurdle
chances of a payment and thereby the claim value decreases; once it falls below the cost of
disutility, the manager does not provide effort any more.

6.2 Continuously Adjusted Risk and Effort Levels

For the stock bonus note that because of condition (29): $\gamma'(0) = \gamma'(\theta_{max}) = c$. Define the critical
asset value $\bar{a}(t) = ce^{-rt}/(\beta \sigma_{max})$. Theorem (12) tells us that the manager will pick maximal risk
and maximal effort for $a > \bar{a}(t)$. For $a < \bar{a}(t)$ she will not provide effort and the risk level is
undetermined.

For the simple bonus, theorem (13) tells us that the manager will pick maximal risk for all $a$.
Furthermore we know from there that the manager does not provide effort for $a < L$, but that
the manager picks maximal effort for $a > L$.

For the capped bonus, theorem (14) tells us that the manager will pick maximal risk for $a > L_2$
and minimal risk for $a < L_1$. Furthermore we know from there that the manager does not provide
effort for $a < L_1$ or $a > L_2$, but that the manager picks maximal effort for $L_1 < a < L_2$.

We now study the bonus-malus model. For $a < L$ the time $T/2$ value of the bonus is zero and
so we conclude based on Theorem (13) that the manager will choose maximal risk and zero effort.
For $a > L$ we use theorem (15) to determine the time $T/2$ value of the bonus; Figure 5 plots for
$a > L$ that value conditional on the time $T/2$ value in the first row as well as its first and second
order derivatives in the second and third row. We see that for $a > L$ the claim value is concave
because the second derivative is negative, but the curvature is fairly small. Note that the claim
value starts at 0 when $a = L$ and increases to approximately 0.0538 at $a = 2$ of which 0.05 come
Figure 4: The time $T/2$ intrinsic bonus function $\tilde{f}$ and its first two derivatives for the bonus-malus scheme.

from the corresponding simple bonus claim. Overall, the claim value at $T/2$ differs only slightly from the simple bonus claim, both in terms of absolute value and in terms of curvature.

Because the time $T/2$ claim value is (locally) concave for $a > L$, theorem 11 tells us the manager’s effort and risk levels. In the definition of the set $S$ the first condition reads $-\theta \frac{(f'(a))^2}{f''(a)} - \gamma' - Ce^{-rt} = -\theta \frac{(f'(a))^2}{f''(a)} - Ce^{-rt}$ gives a zero at $\theta_0 = -Ce^{-rt} \frac{f''(a)}{(f'(a))^2}$; then setting $\sigma_0 = -\theta_0 \frac{f'(a)}{af''(a)} = \frac{Ce^{-rt}}{af'(a)}$

the second condition reads $\sigma_0 \leq \sigma_{max}$.

Figure 4 plots $\theta_0$ and $\sigma_0$ as a function of $a > L$ in the first and second row, respectively for $t = 0$. (For $t > 0$, the levels need to be multiplied by $e^{-rt}$, see our derivation above.) We see that the manager will take risks and provide effort. Effort increases initially, then reaches a
maximum and close to $L$ and then decreases monotonically; risk decreases monotonically as we increase $a > L$. While this is good news, the bad news is that the level of effort and risk is not very large: while the maximal risk level is 0.5, here the manager takes at most 0.01. While the maximal effort level is $\theta_{\max} = 1$, the manager only provides effort less than 0.001.

Overall, the bonus-malus claim value at $T/2$ is similar to the simple bonus, but because it is convex instead of linear as the simple bonus, it induces optimal effort and risk levels less than the maximums. While this is good, the actual size of the optimal risk and effort are deceiving, because the curvature is very small. We are eliminating excessive risk-taking but we are basically eliminating the incentive effects at the same time.

Finally, we look at the bonus scheme developed in subsection 5.4. Figure 6 illustrates the resulting bonus schemes for our parameter choices of the next section, except that we set $\beta = 1\%$ here. Both plots use target effort $\bar{\theta} = 1$; they differ only in the target risk level: the upper plot takes $\bar{\sigma} = 200\%$ so that we are in the case $\frac{d}{\Delta} < 1$; the upper plot takes $\bar{\sigma} = 30\%$ so that we are in the case $\frac{d}{\Delta} > 1$. For the upper plot we calculated $\bar{a}(2) = 1$ and for the lower plot $\bar{a}(2) = 0.9023$. The upper plot resembles a linear payout beyond $\bar{a}(2)$ and is zero below as for the simple bonus;
however, differently to the simple bonus, the bonus function has a slight negative curvature. In the lower plot we see a much stronger negative curvature. Clearly, the situation in the lower plot is the more realistic one, because we can have high target effort ratio, while keeping risk strongly under control. Note that here the size of the bonus payment is not excessive and due to the strong negative curvature the payment increases only as we increase the asset value. (An asset value of 2 corresponds to doubling the initial asset value.)

7 Conclusion

This paper modeled the growth of bank assets and studied the impact of bonus schemes on risk and effort the manager chooses. We showed that single-period and dynamic continuous-time analyses lead to different conclusions regarding managerial behavior. In continuous-time managers follow mostly bang-bang solutions; we motivated this by shrinking the single-period physical time. Because of disappointing risk-implications of current bonus schemes we developed better bonus schemes for which the manager picks a target risk-level. Finally, we illustrated our
Appendix

Proof of Proposition 1. Throughout this proof we assume $a \geq L$. If $a \leq a$, then

$$f_T(\tilde{a}) = \left((1 - \kappa)\beta(a - L)^{+} - \beta(\tilde{a} - a)^{-}\right)^{+} = \beta((1 - \kappa)(a - L) + (\tilde{a} - a))^{+} = \beta(\tilde{a} - L_1(a))^{+} = \beta(\tilde{a} - L_1(a))^{+} - \beta(\tilde{a} - a)^{+}.$$ 

Then, note that $L_1(a) \leq a$ by equation (14). Therefore, if $\tilde{a} > a$, then

$$f_T(\tilde{a}) = \left((1 - \kappa)\beta(a - L)^{+} - \beta(\tilde{a} - a)^{-}\right)^{+} = \beta((1 - \kappa)(a - L))^{+} = \beta((1 - \kappa)(a - L)) - (\tilde{a} - a) = (\tilde{a} - L_1(a))^{+} - (\tilde{a} - a)^{+}.$$ 

This ends the proof. ■

A Constant Risk and Effort Levels

Proof of Theorem 2. We calculate

$$\frac{\partial V_M}{\partial \sigma} = \beta \theta T e^{\theta \sigma T} \text{and} \frac{\partial V_M}{\partial \theta} = \beta \sigma T e^{\theta \sigma T} - \dot{\gamma}'(\theta). \quad (A-1)$$

If the manager provides effort, then $\frac{\partial V_M}{\partial \sigma} > 0$ for all $\sigma$, i.e. the manager picks maximal risk.

Under the condition in the theorem, with maximal risk-level $\frac{\partial V_M}{\partial \theta} > 0$ for all $\theta$. This implies the statement. ■

Proof of Theorem 3. Asymptotically, equation (16) implies that $\dot{\gamma}'(\theta) = \gamma'(\theta)T + O(T^2)$. We have $0 \leq \gamma'(\theta) \leq \gamma'(\theta_{\max})$ for all $0 \leq \theta \leq \theta_{\max}$ because $\gamma'' \geq 0$ by assumption. Therefore, under the condition in the theorem we conclude from equation (A-1) that $\frac{\partial V_M}{\partial \theta} > 0$ for sufficiently small $T$. This implies the statement. ■

Proof of Theorem 4. Based on the value of the manager’s claim, see equations (19, 21, 22), the manager’s first-order conditions w.r.t. the risk-level $\sigma$ are

$$\frac{\partial V_M}{\partial \sigma} = \beta \theta T e^{\theta \sigma T} BS(1, Le^{-\theta \sigma T}, r, \sigma, T) + \beta e^{\theta \sigma T} \left\{ \frac{\partial BS}{\partial \sigma}(1, Le^{-\theta \sigma T}, r, \sigma, T) \frac{\partial Le^{-\theta \sigma T}}{\partial \sigma} + \frac{\partial BS}{\partial \sigma} \right\} = \beta \theta T e^{\theta \sigma T} N \left( d_1(1, Le^{-\theta \sigma T}, r, \sigma, T) \right) + \beta N' \left( d_1(1, Le^{-\theta \sigma T}, r, \sigma, T) \right) \sqrt{T}. \quad (A-2)$$

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The calculation uses the Black-Scholes derivatives that we presented after we introduced the Black-Scholes pricing formula in equation (6) and it uses the derivative \( \frac{\partial L e^{-\theta \sigma T}}{\partial \sigma} = -\theta T L e^{-\theta \sigma T} \). This implies that \( \frac{\partial V_M}{\partial \sigma} \) is always positive, irrespective of the chosen effort-level; therefore, the manager will always head for the maximal risk-level \( \sigma_{max} \).

We calculate for the derivatives of the manager’s value w.r.t. the effort level:

\[
\frac{\partial V_M}{\partial \theta} = -\gamma'(\theta) + \beta \sigma T e^{\theta \sigma T} BS(1, L e^{-\theta \sigma T}, r, \sigma, T) + \beta \sigma T e^{-\gamma T} \mathcal{N}\left(d_2(1, L e^{-\theta \sigma T}, r, \sigma, T)\right)
= -\gamma'(\theta) + \beta \sigma T e^{\theta \sigma T} \mathcal{N}\left(d_1(1, L e^{-\theta \sigma T}, r, \sigma, T)\right)
\]

(A-2)

Because the manager picks the highest risk-level, we need to evaluate this at \( \sigma = \sigma_{max} \). By assumption, \( \frac{\partial V_M}{\partial \theta} \) is then positive for all effort levels \( \theta_{min} \leq \theta \leq \theta_{max} \); so the manager will pick the maximum effort level. ■

**Proof of Theorem 5.** Asymptotically, equation (16) implies that \( \gamma'(\theta) = \gamma'(\theta) T + \mathcal{O}(T^2) \).

From the definition of \( d_1 \), see equation (5), we conclude that

\[
\lim_{T \to 0} d_1(1, L e^{-\theta \sigma_{max} T}, r, \sigma_{max}, T) = \begin{cases} 
\infty & \text{if } L < 1 \\
0 & \text{if } L = 1 \\
-\infty & \text{if } L > 1 
\end{cases}
\]

and so

\[
\lim_{T \to 0} \mathcal{N}\left(d_1(1, L e^{-\theta \sigma_{max} T}, r, \sigma_{max}, T)\right) = \begin{cases} 
1 & \text{if } L < 1 \\
1/2 & \text{if } L = 1 \\
0 & \text{if } L > 1 
\end{cases}
\]

(A-3)

We know from theorem 4 that the manager will pick the highest risk-level. Equation (A-2), evaluated at the risk-level \( \sigma_{max} \) characterizes the first-order derivative of the manager w.r.t. effort. Because \( \gamma'' > 0 \), the assumption in the Theorem implies together with equation (A-3): If \( L \geq 1 \) and \( T \) sufficiently small, \( \frac{\partial V_M}{\partial \theta} > 0 \) for all \( 0 \leq \theta \leq \theta_{max} \); however, if \( L < 1 \) and \( T \) sufficiently small, \( \frac{\partial V_M}{\partial \theta} < 0 \) for all \( 0 \leq \theta \leq \theta_{max} \). ■

**Lemma A-1** For given \( \theta \) define the function

\[
f(\sigma) = \mathcal{N}\left(d_1(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)\right) - \mathcal{N}\left(d_1(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)\right).
\]

If \( \ln(L_1 L_2)/T < 2r \), then the function is always negative. If \( \ln(L_1 L_2)/T \geq 2r \), then the function has a unique zero at \( \sigma_0(\theta) \) defined in equation (24); it is strictly negative for \( \sigma < \sigma_0(\theta) \) and strictly positive for \( \sigma > \sigma_0(\theta) \).
Proof of Lemma A-1. We calculate that
\[
f(\sigma) = \frac{1}{\sqrt{2\pi\sigma^2T}} \exp \left( -\frac{d_1^2(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)}{2} \right) \cdot \left( 1 - \exp \left( \frac{d_1^2(1, L_1 e^{-\theta \sigma T}, r, \sigma, T) - d_1^2(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)}{2} \right) \right).
\]
Using the equality \( \ln^2 L_1 - \ln^2 L_2 = (\ln L_1 + \ln L_2)(\ln L_1 - \ln L_2) = \ln(L_1 L_2) \ln(L_1/L_2) \) we find
\[
d_1^2(1, L_1 e^{-\theta \sigma T}, r, \sigma, T) - d_1^2(1, L_2 e^{-\theta \sigma T}, r, \sigma, T) = \ln(L_1/L_2) \frac{\ln(L_1 L_2) - (2r + 2\theta \sigma + \sigma^2)T}{\sigma^2T},
\]
Because \( L_2 > L_1 \), we have \( \ln(L_1/L_2) < 0 \) and so the sign of \( f \) is the sign of \( \ln(L_1 L_2) - (2r + 2\theta \sigma + \sigma^2)T \). If \( \ln(L_1 L_2)/T < 2r \), this is negative for all \( \sigma \) and so \( f \) is negative for all \( \sigma \). If \( \ln(L_1 L_2)/T \geq 2r \), then for \( \sigma > \sigma_0(\theta) \) (\( \sigma < \sigma_0(\theta) \)) defined in equation (24) this and therefore \( f \) is strictly negative (strictly positive).

Proof of Theorem 6. The positivity statement about the function \( \sigma_0 \) follows directly from the assumption about \( L_1, L_2 \) in relation to \( r, T \). To see that the function \( \sigma_0(\theta) \) is strictly decreasing we calculate the first derivative w.r.t. \( \theta \):
\[
\sigma_0'(\theta) = 1 - \frac{\theta}{\sqrt{\theta^2 + \frac{\ln(L_1 L_2)}{T} - 2r}}.
\]
The term in the square root is greater than \( \theta^2 \) and so the denominator in fraction is larger than \( \theta \), the fraction smaller than one and so \( \sigma_0'(\theta) > 0 \) which proves the statement.

We start analyzing the risk-level for given effort level \( \theta \) for the manager. Because the capped bonus can be interpreted as a long/short portfolio of two simple bonuses, we can use our earlier derived result of equation (A-2) to find that the manager’s first-order derivative w.r.t. the risk-level \( \sigma \) is:
\[
\frac{\partial V_M}{\partial \sigma} = \beta \theta T e^{\theta \sigma T} \left( \mathcal{N}(d_1(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)) - \mathcal{N}(d_1(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)) \right) - \frac{\beta \sqrt{T}}{2} \left( \mathcal{N}^*(d_1(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)) - \mathcal{N}^*(d_1(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)) \right).
\]
The first term is always strictly positive for all \( \sigma \), if \( \theta > 0 \), and zero for all \( \sigma \) if \( \theta = 0 \). The theorem assumes that \( \ln(L_1 L_2)/T \geq r \); Lemma A-1 then implies that the second term is strictly negative for \( \sigma < \sigma_0(\theta) \), zero for \( \sigma = \sigma_0(\theta) \) and strictly positive for \( \sigma > \sigma_0(\theta) \). This implies for \( \theta = 0 \) that the manager will pick the risk-level \( \sigma_M^* = \sigma_0(\theta) \); for \( \theta > 0 \) it implies the manager will pick a risk-level \( \sigma_M^* > \sigma_0(\theta) \), if \( \sigma_0(\theta) < \sigma_{\text{max}} \) and \( \sigma_M^* = \sigma_{\text{max}} \) otherwise.
Next let us look at the effort level of the manager. As before, we can use our earlier derived result of equation (A-2) to calculate for the first-order derivative of the manager’s value w.r.t. effort:

$$\frac{\partial V_M}{\partial \theta} = -\gamma'(\theta) + \beta \sigma T e^{\theta \sigma T} \left( N\left(d_2(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)\right) - N\left(d_2(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)\right) \right). \quad (A-6)$$

The assumption in the theorem states that the term in (A-6) is greater than $\gamma'(\theta)$ for all $\theta$ and all $\sigma$. This implies that $\frac{\partial V_M}{\partial \theta} > 0$ and so the manager will head for maximal effort.  

**Proof of Theorem 7** First we study the manager’s risk-level. We use the characterization of $\frac{\partial V_M}{\partial \sigma}$ in equations (A-4, A-5). As $T$ tends to zero, asymptotically, the second term characterizes $\frac{\partial V_M}{\partial \sigma}$.

If $\ln(L_1 L_2)/T < 2r$, in particular for $1 > L_2 > L_1$, Lemma [A-1] implies that the second term in $\frac{\partial V_M}{\partial \sigma}$ is always negative. Therefore, there is a critical time such that for smaller time $T$ the manager will always pick minimal risk. If $\sigma_{min} = 0$ and the manager picks this risk level, then $\frac{\partial V_M}{\partial \sigma}$ in equation (A-6) is negative for all $\theta$ and the manager decides not to provide effort.

If $\ln(L_1 L_2)/T > 2r$ Lemma [A-1] implies that the second term has a zero at $\sigma_0(\theta)$. Furthermore, there is a critical time such that for smaller time $T$, the term $\sigma_0(\theta) > \sigma_{max}$ for all $\theta$. This implies that the manager will pick maximal risk.

Now we study the manager’s effort level and let us assume the optimal risk-level is greater zero. We then proceed analogous to the proof of theorem 5. Asymptotically, equation (16) implies that $\gamma'(\theta) = \gamma'(\theta) T + O(T^2)$. Furthermore, the limit characterization of $N\left(d_2(1, L e^{-\theta \sigma T}, r, \sigma, T)\right)$ is identical to the limit characterization of $N\left(d_1(1, L e^{-\theta \sigma T}, r, \sigma, T)\right)$ in equation (A-3). This implies that

$$\lim_{T \to 0} N\left(d_2(1, L_1 e^{-\theta \sigma T}, r, \sigma, T)\right) - N\left(d_2(1, L_2 e^{-\theta \sigma T}, r, \sigma, T)\right) = \begin{cases} 
1 & \text{if } L_1 < 1 < L_2 \\
1/2 & \text{if } L_1 = 1, L_2 > 1 \text{ or } L_1 < 1, L_2 = 1 \\
0 & \text{if } L_1 < L_2 < 1 \text{ or } 1 < L_1 < L_2 
\end{cases} \quad (A-7)$$

Because $\gamma'' \geq 0$, the assumption in the Theorem implies together with equations (A-6, A-7) the statement in the theorem about the effort level.  

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Proof of Theorem 8. We define the generator $G$ by setting for $a \geq 0$:

$$ (Gf)(a) = (r + \theta \sigma) af'(a) + \frac{1}{2} \sigma^2 a^2 f''(a). \quad (B-1) $$

Using the Dynkin formula, see Oksendal (1995) we find that the manager’s value is

$$ V_M = f(A_0) + E \left[ \int_0^T (Gf)(A_t) - \gamma(\theta_t) e^{-rt} dt \right] \quad (B-2) $$

This ends the proof. ■

Proof of Theorem 9. We apply theorem 8 using equations (27, 28). Note that $\frac{\partial F}{\partial \sigma} > 0$ for all $0 < \sigma \leq \sigma_{\max}$ and all $0 \leq \theta \leq \theta_{\max}$, and therefore the manager will pick $\sigma^*_M = \sigma_{\max}$. This implies that

$$ \frac{\partial F}{\partial \theta} = \sigma_{\max} af'(a) - \gamma'(\theta) e^{-rt}. $$

Recall that $\gamma'$ is not decreasing. If $\sigma_{\max} af'(a) e^{rt} < \gamma'(0)$, then $\sigma_{\max} af'(a) e^{rt} < \gamma'(\theta)$ and so $\frac{\partial F}{\partial \theta} < 0$ for all $0 \leq \theta \leq \theta_{\max}$; the manager picks $\theta^*_M = 0$. If $\sigma_{\max} af'(a) e^{rt} > \gamma'(\theta_{\max})$, then $\sigma_{\max} af'(a) e^{rt} > \gamma'(\theta)$ and so $\frac{\partial F}{\partial \theta} > 0$ for all $0 \leq \theta < \theta_{\max}$; the manager picks $\theta^*_M = \theta_{\max}$. Finally, if $\gamma'(0) \leq \sigma_{\max} af'(a) e^{rt} \leq \gamma'(\theta_{\max})$, and $\gamma'(0) < \gamma'(\theta_{\max})$ then there is a unique zero to $\frac{\partial F}{\partial \theta}$ which gives $\theta^*_M$ because $\frac{\partial^2 F}{\partial \theta^2} < 0$. If $\gamma'(0) = \sigma_{\max} af'(a) e^{rt} = \gamma'(\theta_{\max})$, then the effort level is undetermined. ■

Proof of Theorem 10. Note that $\gamma' > 0$ by assumption. Therefore, if $f' = 0$, equations (27, 28) imply that the manager will not provide effort and that the risk-level is undetermined. The theorem covers this under the case $\gamma'(0) > 0 = \sigma_{\max} af'(a) e^{rt}$.

We therefore assume $f' > 0$ for the remainder of this proof. We apply theorem 8 note that $F(t, a)$ simplifies to $F(t, a) = (r + \theta \sigma) af'(a) - \gamma(\theta) e^{-rt}$. If the manager provides effort, then clearly with maximal risk level $\sigma_{\max}$; this requires $\gamma'(0) < \sigma_{\max} af'(a) e^{rt}$ and will set the effort level as in the proof of theorem 9. If $\sigma_{\max} af'(a) e^{rt} < \gamma'(0)$ then the manager will not provide any effort and the risk-level is undetermined. If $\sigma_{\max} af'(a) e^{rt} = \gamma'(0)$ then the manager will take maximal risk but the effort level is undetermined. ■

Proof of Theorem 11. We apply theorem 8 using equations (27, 28). First we determine $\sigma_0$ for given $\theta$ such that $\frac{\partial F}{\partial \sigma} = 0$: This defines a function $\sigma_0(\theta) = -\theta f'(a)/(af''(a))$. We then get
for $\sigma = \sigma_0(\theta)$:

$$\frac{\partial F}{\partial \theta} = -\left(\frac{f'(a)}{f''(a)}\right)^2 - \gamma'(\theta)e^{-rt}.$$  

Note that the first term makes a positive contribution, while the second makes a negative contribution. Therefore, if this term has a zero for some $\theta$, but the resulting $\sigma_0$ is larger than $\sigma_{\text{max}}$, then can only pick the risk level $\sigma_{\text{max}}$, the positive contribution from the first term will be smaller and this is no longer a zero for that $\theta$.

If the set $S_t$ is empty, then $\frac{\partial F}{\partial \theta}$ has the sign of $\frac{\partial F}{\partial \theta}(0) = -\gamma'(0)e^{-rt}$, i.e. it is always negative. Therefore the effort level is then $\theta^*_M = 0$. This implies $\frac{\partial F}{\partial \sigma}(a) = \sigma a^2 f''(a) < 0$ for all $\sigma$ and so the manager picks $\sigma^*_M = \sigma_{\min}$.

If the set $S_t$ is not empty, then any zero is a potential maximum and we pick that one which maximizes $F$; the risk level is then the corresponding $\sigma_0$.

**Proof of Theorem 13.**  The simple bonus function is twice continuously differentiable except at the kink at the strike $L$. For any $\epsilon > 0$ we do a local smooth pasting by dividing the time interval $[0, T]$ into the interval $[0, T - \epsilon]$ and the interval $[T - \epsilon, T]$ and assume that over the second interval the manager and the owner desire no effort but maximal risk. The bonus claim value is then $f_\epsilon(a) = \beta BS(a, L, r, \sigma_{\text{max}}, T - \epsilon)$. We have

$$f_\epsilon'(a) = \beta N\left(d_1(a, L, r, \sigma_{\text{max}}, T - \epsilon)\right), \quad f_\epsilon''(a) = \frac{\beta N'(d_1(a, L, r, \sigma_{\text{max}}, T - \epsilon))}{a \sigma_{\text{max}} \sqrt{\epsilon}}.$$  

Let us first analyze the situation when $a < L$. The function has the property that $f_\epsilon'(a)$ and $f_\epsilon''(a)$ are both positive and converge monotonically to zero as $\epsilon$ tends to 0. Then in equation (27) $\frac{\partial F}{\partial \theta}(a) < 0$ and so for sufficiently small $\epsilon$ the manager will pick no effort $\theta^*_M(t, a) = 0$; equation (28) then implies $\frac{\partial F}{\partial \sigma}(a) > 0$ and so for sufficiently small $\epsilon$ the manager will pick maximal risk $\sigma^*_M(t, a) = \sigma_{\text{max}}$.

Finally let us look at the situation when $a < L$. The function has the property that $f_\epsilon'(a)$ is monotonically increasing and converges to 1 as $\epsilon$ tends to 0; the second derivative $f_\epsilon''(a)$ is positive and converges monotonically to zero as $\epsilon$ tends to 0. Then in equation (28) $\frac{\partial F}{\partial \sigma}(a) > 0$ and so for sufficiently small $\epsilon$ the manager will pick maximal risk $\sigma^*_M(t, a) = \sigma_{\text{max}}$; for sufficiently small $\epsilon$ equation (28) reads

$$\frac{\partial F}{\partial \sigma}(a) = \sigma_{\text{max}} a - \gamma'(\theta)e^{-rt}.$$  

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We then proceed for the manager as in the proof of theorem 9, setting $f' = 1$. We have show the theorem for all value $a$ except at the kink $L$; however this is a single event of probability measure zero.

**Proof of Theorem 14.** Similar to the proof of theorem 13 we first define the function $f_\epsilon(a) = \beta BS(a, L_1, r, \sigma_{max}, T - \epsilon) - \beta BS(a, L_2, r, \sigma_{max}, T - \epsilon)$ as the claim value in a Black-Scholes world without effort and maximal risk. First we study the limit properties of the first derivative and calculate:

$$f'_\epsilon(a) = \beta N\left(d_1(a, L_1, r, \sigma_{max}, T - \epsilon)\right) - \beta N\left(d_1(a, L_2, r, \sigma_{max}, T - \epsilon)\right),$$

For $a < L_1$ and $a > L_2$ we find that $f'_\epsilon(a)$ is positive and converges monotonically to zero as $\epsilon$ tends to 0. For $L_1 < a < L_2$ we find that $f'_\epsilon(a)$ converges monotonically to $a - L_1$ as $\epsilon$ tends to 0.

Next we study the limit properties of the second derivative and calculate:

$$f''_\epsilon(a) = \frac{N\left(d_1(a, L_1, r, \sigma_{max}, T - \epsilon)\right) - N\left(d_1(a, L_2, r, \sigma_{max}, T - \epsilon)\right)}{a \sigma_{max} \sqrt{\epsilon}}.$$

We know that $f''_\epsilon(a)$ tends to zero for all $a$ except $L_1, L_2$. However, we need to know the sign and for this we calculate that the numerator in $f''_\epsilon(a)$ is equal to

$$\frac{1}{\sqrt{2\pi \sigma_{max}^2 T}} \exp\left(-\frac{d_1^2(a, L_1, r, \sigma_{max}, T)}{2}\right) \cdot \left(1 - \exp\left(\frac{d_1^2(a, L_1, r, \sigma_{max}, T) - d_1^2(a, L_2, r, \sigma_{max}, T)}{2}\right)\right).$$

We calculate that

$$d_1^2(a, L_1, r, \sigma_{max}, T) - d_1^2(a, L_2, r, \sigma_{max}, T) = \ln\left(\frac{a^2}{(L_1 L_2)}\right) \frac{\ln(L_2/L_1) - (2r + \sigma^2)T}{\sigma^2 T},$$

where we used the equality $\ln^2(a/L_1) - \ln^2(a/L_2) = (\ln(a/L_1) + \ln(a/L_2))(\ln(a/L_1) - \ln(a/L_2)) = \ln(a/(L_1 L_2)) \ln(L_2/L_1)$. This implies that the sign is positive as long as $a < \sqrt{L_1 L_2}$ and negative otherwise.

For $a < \sqrt{L_1 L_2}$ we proceed exactly as in the proof of theorem 13 to conclude the statement; for $a > \sqrt{L_1 L_2}$ we note that $-f'_\epsilon(a)/(a f''_\epsilon(a))$ tends to infinity to conclude the statement. ■

**Proof of Theorem 15.** Here we analyze the time $T/2$ value of the deferred bonus. For this we denote by $a = A_{T/2}$ the conditional time $T/2$ asset value. For $a < L$ the bonus value is
zero and therefore we assume that \( A_{T/2} = a > L \). Section 3 characterized the resulting time \( T \) payment as a capped bonus scheme with lower hurdle \( L_1(a) = a - (1 - \kappa) \cdot (a - L)^+ \) and upper hurdle \( L_2 = a \). We denote \( \tilde{f}(a) \) the time \( T/2 \) value from this claim; using the Dynkin formula as in the proof Theorem 8 and using the smooth pasting approximation of Theorem 14 we find the analogous formula to equation (B-2) over the time interval \( T/2 \) to \( T \):

\[
\tilde{f}(a) = f(a) + E \left[ \int_{T/2}^{T} (Gf) \left( a \frac{A_t}{A_{T/2}} \right) - \gamma(\theta_t) e^{-r(t-T/2)} dt \right]. \tag{B-3}
\]

We define

\[
g(a) = E \left[ \int_{T/2}^{T} (Gf) \left( a \frac{A_t}{A_{T/2}} \right) - \gamma(\theta_t) e^{-r(t-T/2)} dt \right], \tag{B-4}
\]

and will now characterize the function \( g \). We know from our earlier analysis of the capped bonus in the proof of theorem 14 that \( f'(a) = \beta \) for \( L_1 \leq a \leq L_2 \), \( f'(a) = 0 \) for \( a < L_1 \), and for \( a > L_2 \) and that \( f'' = 0 \) everywhere. (This is the limit result of the function under the smooth pasting procedure.) Furthermore, we know that the owner picks \( \theta_{max}, \sigma_{max} \) between \( L_1 \) and \( L_2 \). Therefore we have based on equation (B-1):

\[
(Gf) \left( a \frac{A_t}{A_{T/2}} \right) = \begin{cases} 
\beta(r + \theta_{max}\sigma_{max})a \frac{A_t}{A_{T/2}} & \text{if } L_1 < a \frac{A_t}{A_{T/2}} < L_2 \\
0 & \text{otherwise}
\end{cases}
\]

This implies

\[
g_1(a) = \frac{1}{\beta} E \left[ \int_{T/2}^{T} (Gf) \left( a \frac{A_t}{A_{T/2}} \right) dt \right] = E \left[ \int_{T/2}^{T} a \frac{A_t}{A_{T/2}} 1_{L_1 < a \frac{A_t}{A_{T/2}} < L_2} dt \right] = \int_{T/2}^{T} e^{r(T-t/2)} E \left[ e^{-r(t-T/2)} a \frac{A_t}{A_{T/2}} 1_{L_1 < a \frac{A_t}{A_{T/2}} < L_2} \right] dt
\]

\[
= \int_{T/2}^{T} e^{r(T-t/2)} \left\{ \mathcal{N} \left( d_1 \left( a, L_1, r, \sigma_{max}, t - \frac{T}{2} \right) \right) - \mathcal{N} \left( d_1 \left( a, L_2, r, \sigma_{max}, t - \frac{T}{2} \right) \right) \right\} dt,
\]

where the third equality follows from the derivation of the Black-Scholes formula. Similarly we find that

\[
g_2(a) = \frac{1}{\beta} E \left[ \int_{T/2}^{T} \gamma(\theta_t) e^{-r(t-T/2)} dt \right] = E \left[ \int_{T/2}^{T} \gamma(\theta_{max}) e^{-r(t-T/2)} 1_{L_1 < a \frac{A_t}{A_{T/2}} < L_2} dt \right]
\]

\[
= \gamma(\theta_{max}) \int_{T/2}^{T} e^{-r(t-T/2)} E \left[ 1_{L_1 < a \frac{A_t}{A_{T/2}} < L_2} \right] dt
\]

\[
= \gamma(\theta_{max}) \int_{T/2}^{T} \left\{ \mathcal{N} \left( d_2 \left( a, L_1, r, \sigma_{max}, t - \frac{T}{2} \right) \right) - \mathcal{N} \left( d_2 \left( a, L_2, r, \sigma_{max}, t - \frac{T}{2} \right) \right) \right\} dt.
\]
This implies that the function \( g_1, g_2 \) take the stated form and together with equation (B-4) that
the total time \( T/2 \) value function (the paid out bonus plus the value of the deferred bonus) is
\[ \tilde{f}(a) = 1_{a>L} \{ \beta(a-L) + g_1(a) - g_2(a) \}, \]
as stated.

**Proof of Theorem 16.** We proceed as in the proof of theorem 8 applying Dynkin’s
formula with the function \( \tilde{f}(a) = a - f(a) \) but without cost of disutility \( \gamma \). We find that we need
to maximize
\[(G\tilde{f})(v) = (r + \theta \sigma) a \tilde{f}'(a) + \frac{1}{2} \sigma^2 a^2 \tilde{f}''(a)\]
Note that \( \tilde{f}' = 1 - f' \), \( \tilde{f}'' = f'' \). The first- and second order derivatives w.r.t. \( \theta \) are then
\[\frac{\partial (Gf)}{\partial \theta} = \theta \sigma a \tilde{f}'(a), \quad \frac{\partial^2 (Gf)}{\partial \theta^2} = 0\]
This implies the the owner will desire maximal effort. The first- and second order derivatives
w.r.t. \( \sigma \) are
\[\frac{\partial (Gf)}{\partial \sigma} = \theta a(1 - f'(a)) + \sigma a^2 f''(a), \quad \frac{\partial^2 (Gf)}{\partial \sigma^2} = a^2 f''(a)\]
Note that \( \frac{\partial (Gf)}{\partial \sigma} > 0 \) at \( \sigma = 0 \), because \( \theta^* = \theta_{\text{max}} \). So \( \sigma_{\text{min}} \) can never be optimal.

If \( f''(a) > 0 \) the first derivative is always positive and so the owner will desire the maximal
risk-level.

If \( f''(a) < 0 \), then note that while \( \frac{\partial (Gf)}{\partial \sigma} > 0 \) at \( \sigma = 0 \), \( \frac{\partial (Gf)}{\partial \sigma} \) is decreasing in \( \sigma \) because
\( f''(a) < 0 \). Furthermore, there is a single zero at \( -\theta_{\text{max}} \frac{1-f'(a)}{a f''(a)} > 0 \). If this is less than \( \sigma_{\text{max}} \), this
is a maximum for the owner’s value because \( f''(a) < 0 \); otherwise, if this is larger than \( \sigma_{\text{max}} \),
then we know that \( \frac{\partial (Gf)}{\partial \sigma} > 0 \) for all \( \sigma_{\text{min}} \leq \sigma \leq \sigma_{\text{max}} \) and so the maximum value for the owner
is at the maximal risk-level. ■

**Proof of Theorem 17.** For the stock bonus scheme, the owner’s value is \( (1 - \beta) \exp(\theta \sigma T) \)
which is maximized for maximal effort and risk levels.

For the simple bonus scheme we proceed similar to the proof of theorem 13 but looking at
limits of \( 1 - f_\epsilon \) instead of \( f_\epsilon \) as \( \epsilon \) to zero; we then find through theorem 16 that the owner will
desire maximal risk and effort levels \( \theta_{\tilde{\sigma}}(a) = \theta_{\text{max}}, \sigma_{\tilde{\sigma}}(a) = \sigma_{\text{max}} \).

For the capped bonus scheme, we proceed as in the proof of theorem 14. ■

**Proof of Theorem 18.** For \( \tilde{\sigma} \) to be the optimal risk-level we conclude based on equation
(28) that we need \( \frac{\partial F}{\partial \sigma} = 0 \) for \( \sigma = \tilde{\sigma} \). This implies
\[- \frac{f'(a)}{f''(a)} = \tilde{\sigma} \theta a. \]
This is an equation that is similar to the equation on linear risk-tolerance in utility theory. We know from there that $f$ needs to be a power function with exponent $1 - \frac{\bar{\theta}}{\bar{\sigma}}$. If $\frac{\bar{\sigma}}{\bar{\theta}} < 1$, then for some $\beta > 0$ we must have

$$f(a) = \beta a^{1 - \frac{\bar{\theta}}{\bar{\sigma}}}, \text{ so that } f'(a) = \beta \left(1 - \frac{\bar{\theta}}{\bar{\sigma}}\right) a^{-\frac{\bar{\theta}}{\bar{\sigma}}}, f''(a) = -\beta \left(1 - \frac{\bar{\theta}}{\bar{\sigma}}\right) \frac{\bar{\theta}}{\bar{\sigma}} a^{-\frac{\bar{\theta}}{\bar{\sigma}} - 1}.$$

This implies that $f(a) \geq 0, f'(a) \geq 0, f''(a) < 0$ for all $a > 0$; so it defines a bonus function and $\bar{\sigma}$ is the optimal risk level. If $\frac{\bar{\sigma}}{\bar{\theta}} < 1$, then to ensure that the first derivative is nonnegative, we must take the negative of $f$, i.e. for some $\beta > 0$ we must have

$$f(a) = -\beta a^{1 - \frac{\bar{\theta}}{\bar{\sigma}}},$$

so that $f' \geq 0, f'' < 0$ and $\bar{\sigma}$ is the optimal risk level. ■

**Proof of Theorem 19.** We study only $\frac{\bar{\sigma}}{\bar{\theta}} < 1$, the case $\frac{\bar{\sigma}}{\bar{\theta}} > 1$ follows analogously. Using the risk level $\bar{\sigma}$ and the form of the derivatives of the bonus function derived in the proof of theorem 18, equation (27) reads

$$\frac{\partial F}{\partial \theta} = \beta \bar{\sigma} \left(1 - \frac{\bar{\theta}}{\bar{\sigma}}\right) a^{1 - \frac{\bar{\theta}}{\bar{\sigma}}} - \gamma'(\theta)e^{-rt}.$$

We calculate that $\frac{\partial F}{\partial \theta}$ is zero for $a = \bar{a}(t)$, and that it is positive (negative) for all $a > \bar{a}(t)$ ($a < \bar{a}(t)$). This implies the statement. ■

**References**


